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GALERKIN APPROXIMATIONS FOR THE STOCHASTIC BURGERS EQUATION

DIRK BLÖMKER* AND ARNULF JENTZEN†

Abstract. Existence and uniqueness for semilinear stochastic evolution equations with additive noise by means of finite dimensional Galerkin approximations is established and the convergence rate of the Galerkin approximations to the solution of the stochastic evolution equation is estimated.

These abstract results are applied to several examples of stochastic partial differential equations (SPDEs) of evolutionary type including the stochastic heat equation, stochastic reaction diffusion equations and the stochastic Burgers equation. The estimated convergence rates are illustrated by numerical simulations.

The main novelty in this article is to estimate the difference of the finite dimensional Galerkin approximations and of the infinite dimensional SPDE uniformly in space, i.e. in the L^∞-topology, instead of the usual Hilbert-space estimates in the L^2-topology, that were shown before.

Key words. Galerkin approximations, stochastic partial differential equation, stochastic heat equation, stochastic reaction diffusion equation, stochastic Burgers equation, strong error criteria.

AMS subject classifications. 60H15, 35K90

1. Introduction. In this work we present a very general result for the spatial approximation of stochastic evolution equations with additive noise via Galerkin methods. This abstract result is applied to several examples of stochastic partial differential equations (SPDEs) of evolutionary type including the stochastic heat equation, stochastic reaction diffusion equations and the stochastic Burgers equation. In all examples we need to verify the following conditions. First we need the rate of approximation of the linear equation obtained by omitting the nonlinear term in the stochastic evolution equation. Then one needs a quite weak local Lipschitz conditions for the nonlinearity, and finally a uniform bound on the sequence of approximations. These results are the key for the main theorem (see Theorem 3.1). The main novelty in this article is to estimate the difference of the finite dimensional Galerkin approximations and of the infinite dimensional SPDE uniformly in space, i.e. in the L^∞-topology, instead of the usual Hilbert-space estimates in the L^2-topology, that were shown before.

Although there are several different methods using finite dimensional approximations like, for instance, spectral Galerkin, finite elements, or wavelets, we focus here on the spectral Galerkin method in all our examples. Thus the finite dimensional approximations are given by an expansion in terms of the eigenfunctions of a dominant linear operator. This spectral Galerkin approximation is one of the key tools in the analysis of stochastic or deterministic PDEs. For SPDEs see for example [8, 4, 9, 2], where the Galerkin method was used to establish the existence of solutions. Moreover, spectral methods are an effective tool for numerical simulations, especially on domains, like the interval, where fast Fourier-transforms are available. Nevertheless, it is limited on domains, where the eigenfunctions of the dominant linear operator are not explicitly known. In recent years there has also been a significant interest in

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analytic results for the rate of approximation using a spectral Galerkin method as a numerical method. Some examples are [10, 20, 21, 17, 22, 23, 24, 25, 15, 19], where in some cases also the full discretization is treated including the time discretization.

In order to illustrate the main result of this article we limit ourself in this introductory section to the stochastic Burgers equation with Dirichlet boundary conditions and refer to Section 3 for the general result and to Section 4 for further examples. To this end let $T > 0$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space, and let $X : [0,T] \times \Omega \to C([0,1], \mathbb{R})$ be the up to indistinguishability unique solution of the SPDE

$$dX_t = \left[\frac{\partial^2}{\partial x^2} X_t - X_t \frac{\partial}{\partial x} X_t\right] dt + dW_t, \qquad X_t(0) = X_t(1) = 0, \qquad X_0 = 0 \tag{1.1}$$

for $t \in [0,T]$ and $x \in (0,1)$, where W_t, $t \in [0,T]$, is a cylindrical Wiener process, which models space-time white noise on $(0,1)$. The initial value $X_0 = 0$ is here zero for simplicity of presentation. The existence and uniqueness of solutions of the stochastic Burgers equation was studied by Nualart & Gyöngy [12] on the whole real line, and by Da Prato, Debussche & Temam [5] with Dirichlet boundary conditions on the interval (see also [7]).

Recently, Gyöngy & Alabert showed the following error estimate for spatial discretizations in the L^2-topology (see Theorem 2.2 in [1]):

$$\mathbb{P}\left[\sup_{0\le t\le T}\left(\int_0^1 \left|X_t(x) - X_t^N(x)\right|^2 dx\right)^{\frac{1}{2}} \le C_\varepsilon \cdot N^{\left(\frac{1}{2}-\varepsilon\right)}\right] = 1 \tag{1.2}$$

for every $N \in \mathbb{N} := \{1, 2, \dots\}$ and every arbitrarily small $\varepsilon \in (0, \frac{1}{2})$ with random variables $C_\varepsilon : \Omega \to [0, \infty)$, $\varepsilon \in (0, \frac{1}{2})$, where the X^N, $N \in \mathbb{N}$, are given by finite differences approximations.

The main result in this article (Theorem 3.1) applied to equation (1.1) (see Section 4.3) yields the following estimate:

$$\mathbb{P}\left[\sup_{0\le t\le T}\sup_{0\le x\le 1}\left|X_t(x) - X_t^N(x)\right| \le C_\varepsilon \cdot N^{\left(\frac{1}{2}-\varepsilon\right)}\right] = 1 \tag{1.3}$$

for every $N \in \mathbb{N}$ and every arbitrarily small $\varepsilon \in (0, \frac{1}{2})$ with random variables $C_\varepsilon : \Omega \to [0, \infty)$, $\varepsilon \in (0, \frac{1}{2})$, where X^N, $N \in \mathbb{N}$, are spectral Galerkin approximations. Thus, although the spatial error criteria is estimated in the bigger L^∞-norm instead of the L^2-norm, the convergence rate remains $\frac{1}{2}-$. This convergence rate with respect to the strong L^∞-norm is also corroborated by numerical examples (see Section 4). (For a real number $a > 0$, we write $a-$ for the convergence order, if the convergence order is better than $a - \varepsilon$ for every arbitrarily small $\varepsilon \in (0, a)$.)

Another related result is given by Liu [21]. He treats stochastic reaction diffusion equations of the Ginzburg-Landau type which fit in our abstract setting. For such equations he obtained estimates in the H^r-topology with the rate $\left(\frac{1}{2} - r\right) -$ for every $r \in (0, \frac{1}{2})$. This also yields estimates in the L^p-topology. Nevertheless, such estimates do not yield convergence in the L^∞-topology, since in one dimension H^r is embedded into L^∞ for $r > \frac{1}{2}$ only. Moreover, in contrast to (1.3) this would not give a convergence rate $\frac{1}{2}-$ in any L^p-topology, $p \in (2, \infty]$.

Having indicated the results of this article we now illustrate how the improvement concerning the L^∞-topology could be achieved. To this end we write equation (1.1) in the mild solution form

$$X_t = \int_0^t e^{A(t-s)} F(X_s)\, ds + \int_0^t e^{A(t-s)} dW_s \qquad \mathbb{P}-\text{a.s.}$$

for all $t \in [0,T]$, where A is the Laplacian with Dirichlet boundary conditions on $(0,1)$ and where $F(v) = -\frac{1}{2}\partial_x(v^2)$ for every $v \in C([0,1],\mathbb{R})$ (see Section 4.3 for a detailed description of the nonlinearity, in particular for the vector spaces involved in order to define the nonlinearity of the stochastic Burgers equation). The key estimate in order to establish (1.3) is to show

$$\sup_{0\le t\le T} \sup_{0\le x\le 1} |X_t(x) - P_N X_t(x)| \le C_\varepsilon \cdot N^{(\frac{1}{2}-\varepsilon)} \qquad \mathbb{P}-\text{a.s.}$$

for every $N \in \mathbb{N}$ and every $\varepsilon \in (0,\frac{1}{2})$ with appropriate random variables $C_\varepsilon : \Omega \to [0,\infty)$, $\varepsilon \in (0,\frac{1}{2})$, and where $P_N : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is the standard Galerkin projection given by

$$(P_N v)(x) = \sum_{n=1}^{N} 2\sin(n\pi x) \int_0^1 \sin(n\pi y)\, v(y)\, dy$$

for every $x \in [0,1]$, $v \in C([0,1],\mathbb{R})$ and $N \in \mathbb{N}$. For clarity of exposition we omit in the following the supremum in time and illustrate the estimate

$$\|X_T - P_N X_T\|_V \le C_\varepsilon \cdot N^{(\frac{1}{2}-\varepsilon)} \qquad \mathbb{P}-\text{a.s.}$$

for every $N \in \mathbb{N}$ and every $\varepsilon \in (0,\frac{1}{2})$, where $\|v\|_V := \sup_{0\le x\le 1} |v(x)|$. for $v \in C([0,1],\mathbb{R})$.

The spatial discretization error is often measured via estimates of the following form (see e.g. [10, 20, 21]):

$$\begin{aligned} \|X_T - P_N X_T\|_H &= \|(I-P_N)X_T\|_H \\ &\le \|(-A)^{-r}(I-P_N)\|_{L(H)} \cdot \|(-A)^r X_T\|_H \le N^{-2r} \cdot \|(-A)^r X_T\|_H \end{aligned} \tag{1.4}$$

for every $r \in (0,\frac{1}{4})$ and every $N \in \mathbb{N}$, where $\|v\|_H := (\int_0^1 |v(y)|^2 dy)^{\frac{1}{2}}$ is the norm in $H := L^2((0,1),\mathbb{R})$. In this way the convergence order $\frac{1}{2}-$ for the spatial discretization error in the L^2-topology can be achieved. Since the Sobolev embedding

$$\|v\|_V \le D_r \|(-A)^r v\|_H \quad \text{for every } v \in D((-A)^r)$$

with appropriate constants $D_r > 0$ nevertheless holds for $r > \frac{1}{4}$ only (see e.g. Section 4 in [14]), estimates of the form (1.4) seem not to be an adequate instrument for deriving estimates for the spatial discretization error in the finer L^∞-topology. Instead of (1.4) we therefore use the following estimates here:

$$\begin{aligned} \|X_T - P_N X_T\|_V &= \|(I-P_N)X_T\|_V \\ &\le \left\|(I-P_N)\int_0^T e^{A(T-s)} F(X_s)\, ds\right\|_V + \left\|(I-P_N)\int_0^T e^{A(T-s)}\, dW_s\right\|_V \end{aligned} \tag{1.5}$$

for every $N \in \mathbb{N}$. Then, the first summand on the right hand side of (1.5) can be estimated by classical Sobolev embeddings, i.e. (cf. (1.4))

$$
\begin{aligned}
\left\|(I - P_N) \int_0^T e^{A(T-s)} F(X_s)\, ds\right\|_V &\le D_r \left\|(-A)^r (I - P_N) \int_0^T e^{A(T-s)} F(X_s)\, ds\right\|_H \\
&\le D_r \left\|(-A)^{-\rho} (I - P_N)\right\|_{L(H)} \left\|(-A)^{(\rho+r)} \int_0^T e^{A(T-s)} F(X_s)\, ds\right\|_H \\
&\le D_r\, N^{-2\rho} \left\|(-A)^{\rho+r} \int_0^T e^{A(T-s)} F(X_s)\, ds\right\|_H
\end{aligned}
$$

for every $N \in \mathbb{N}$ and every $r > \frac{1}{4}$, $\rho > 0$ with $r + \rho < 1$. Indeed, this is possible, since $\int_0^T e^{A(T-s)} F(X_s)\, ds$ is spatially much smoother than the original solution X_T.

For the second term on the right hand side of (1.5) we strongly exploit the fact that the discrete stochastic process $(Z_N)_{N\in\mathbb{N}}$ in $C([0,1], \mathbb{R})$ defined by

$$
Z_N \;:=\; P_N \int_0^T e^{A(T-s)} dW_s \;=\; \sum_{n=1}^{N} \sqrt{2} \sin(n\pi \cdot) \int_0^T e^{-n^2\pi^2(T-s)} d\beta_s^n \qquad \mathbb{P} - \text{a.s.}
$$

with appropriate independent standard Brownian motions β^n, $n \in \mathbb{N}$, has independent normal distributed increments and is in particular a Gaussian martingale in the space of continuous functions. Moreover, the stochastic convolution $\int_0^T e^{A(T-s)} dW_s$ is the limit of the discrete martingale $(Z_N)_{N\in\mathbb{N}}$ as $N \to \infty$. These observations enable us to obtain sharp estimates for

$$
\left\|(I - P_N) \int_0^T e^{A(T-s)} dW_s\right\|_V = \left\| \int_0^T e^{A(T-s)} dW_s - Z_N\right\|_V \tag{1.6}
$$

for every $N \in \mathbb{N}$ by exploiting the independence of the increments of $(Z_N)_{N\in\mathbb{N}}$ and bounds on the eigenfunctions of the Laplacian (see Proposition 4.2 below).

While in other contexts such as regularity analysis of the stochastic convolution $\int_0^t e^{A(t-s)}\, dW_s$, $t \in [0, T]$, related estimates are often used (see e.g. Theorem 5.20 in [6] or Proposition 1.1 and Proposition 1.2 in [4]), this approach seems to be new for the estimation of the spatial discretization error.

To sum up the key idea to obtain estimates for the spatial discretization error in the L^∞-topology is to divide the SPDE into a random PDE part and a Gaussian martingale part (a Gaussian martingale with respect to the projections $(P_N)_{N\in\mathbb{N}}$!) and then to apply classical Sobolev embeddings to the random PDE part and Gaussian martingale methods to the martingale part (see Theorem 3.1).

Finally, we would like to comment on the importance of estimates in the L^∞-topology. On the one hand (1.3) is simply a stronger assertion then (1.2) since the convergence rate is $\frac{1}{2}-$ in both cases. On the other hand to show convergence of full discretizations of SPDEs with non-globally Lipschitz coefficients such as the stochastic Burgers equation it is roughly speaking necessary to have estimates in the L^∞-topology which can be seen in the instructive results of Gyöngy (see Theorem 4.2 in [11]) and Petterson & Signahl (see Theorem 3.1 in [26]).

The rest of the paper is organized as follows. Section 2 gives the setting and the assumptions for the main result, which is then presented in Section 3. In Section 4 we discuss our examples, while in the final Section, we state most of the proofs.

2. Setting and assumptions. Throughout this article suppose that the following setting and the following assumptions are fulfilled. The first assumption is a regularity and approximation condition on the semigroup of the linear operator. The second is a local Lipschitz condition on the nonlinearity. The third is an assumption on the approximation of the stochastic convolution, while the final one is a uniform bound (in the approximation rate N) on the finite dimensional approximations.

Fix now $T > 0$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two $\mathbb{R}$-Banach spaces. Moreover, let $P_N : V \to V$, $N \in \mathbb{N}$, be a sequence of bounded linear operators from V to V.

Assumption 1 (Semigroup S). *Let $S : (0, T] \to L(W, V)$ be a continuous mapping satisfying*

$$\sup_{0<t\le T} \left(t^{\alpha} \|S_t\|_{L(W,V)} \right) < \infty, \qquad \sup_{N\in\mathbb{N}} \sup_{0<t\le T} \left(t^{\alpha} N^{\gamma} \|S_t - P_N S_t\|_{L(W,V)} \right) < \infty,$$

where $\alpha \in [0, 1)$ and $\gamma \in (0, \infty)$ are given constants.

Assumption 2 (Nonlinearity F). *Let $F : V \to W$ be a mapping, which satisfies*

$$\sup_{\substack{\|v\|_V, \|w\|_V \le r, \\ v \ne w}} \frac{\|F(v) - F(w)\|_W}{\|v - w\|_V} < \infty$$

for every $r > 0$.

Assumption 3 (Stochastic process O). *Let $O : [0, T] \times \Omega \to V$ be a stochastic process with continuous sample paths and $\sup_{N\in\mathbb{N}} \sup_{0\le t\le T} N^{\gamma} \|O_t(\omega) - P_N (O_t(\omega))\|_V < \infty$ for every $\omega \in \Omega$, where $\gamma \in (0, \infty)$ is given in Assumption 1.*

Assumption 4 (Existence of solutions). *Let $X^N : [0, T] \times \Omega \to V$, $N \in \mathbb{N}$, be a sequence of stochastic processes with continuous sample paths and with*

$$X_t^N(\omega) = \int_0^t P_N\, S_{(t-s)}\, F(X_s^N(\omega))\, ds + P_N\left(O_t(\omega)\right), \quad \sup_{M\in\mathbb{N}} \sup_{0\le s\le T} \|X_s^M(\omega)\|_V < \infty \tag{2.1}$$

for every $t \in [0, T]$, $\omega \in \Omega$ and every $N \in \mathbb{N}$.

We call here a mapping $Y : [0, T] \times \Omega \to V$ a stochastic process, if the mappings

$$Y_t : \Omega \to V, \qquad \omega \mapsto Y_t(\omega) := Y(t, \omega), \qquad \omega \in \Omega$$

are $\mathcal{F}/\mathcal{B}(V)$-measurable for every $t \in [0, T]$. Additionally, we say that a stochastic process $Y : [0, T] \times \Omega \to V$ has continuous sample paths, if the mappings

$$[0, T] \to V, \qquad t \mapsto Y_t(\omega), \qquad t \in [0, T]$$

are continuous for every $\omega \in \Omega$. Moreover, note that if $Y : [0, T] \times \Omega \to V$ is a stochastic process with continuous sample paths, then the V-valued Bochner integral

$$\int_0^t P_N\, S_{(t-s)}\, F(Y_s(\omega))\, ds$$

in Assumption 4 is well defined for every $\omega \in \Omega$, $t \in [0, T]$ and every $N \in \mathbb{N}$ due to Assumption 1 and Assumption 2.

3. Main result. In this section we state and prove the main approximation result, which is based on the assumptions of the previous section.

THEOREM 3.1. *Let Assumptions 1-4 be fulfilled. Then, there exists a unique stochastic process $X : [0,T] \times \Omega \to V$ with continuous sample paths, which fulfills*

$$X_t(\omega) = \int_0^t S_{(t-s)}\, F(X_s(\omega))\, ds + O_t(\omega) \tag{3.1}$$

for every $t \in [0,T]$ and every $\omega \in \Omega$. Moreover, there exists a $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mapping $C : \Omega \to [0,\infty)$ such that

$$\sup_{0 \le t \le T} \left\| X_t(\omega) - X_t^N(\omega) \right\|_V \le C(\omega) \cdot N^{-\gamma}$$

holds for every $N \in \mathbb{N}$ and every $\omega \in \Omega$, where $\gamma \in (0,\infty)$ is given in Assumption 1.

Proof. [Proof of Theorem 3.1] Consider the $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mapping $R : \Omega \to [0,\infty)$ given by

$$\begin{aligned}
R(\omega) &:= \sup_{N \in \mathbb{N}} \sup_{0 \le t \le T} \left\| F(X_t^N(\omega)) \right\|_W + T + \sup_{N \in \mathbb{N}} \sup_{0 \le t \le T} \left(N^\gamma \left\| O_t(\omega) - P_N\left(O_t(\omega)\right) \right\|_V \right) \\
&+ \frac{1}{1-\alpha} + \sup_{N \in \mathbb{N}} \sup_{0 < t \le T} \left(t^\alpha \left\| P_N S_t \right\|_{L(W,V)} \right) + \sup_{N \in \mathbb{N}} \sup_{0 < t \le T} \left(t^\alpha N^\gamma \left\| S_t - P_N S_t \right\|_{L(W,V)} \right)
\end{aligned}$$

for every $\omega \in \Omega$, which is finite due to Assumptions 1-4. Note that R is indeed $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable, although V is not assumed to be separable! Additionally, we consider the $\mathcal{B}([0,\infty))/\mathcal{B}([0,\infty))$-measurable mapping $L : [0,\infty) \to [0,\infty)$ given by

$$L(r) := \sup_{\substack{\|v\|_V, \|w\|_V \le r \\ v \ne w}} \frac{\left\| F(v) - F(w) \right\|_W}{\left\| v - w \right\|_V}$$

for every $r \in [0,\infty)$ and the $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mapping $Z : \Omega \to [0,\infty)$ given by

$$Z(\omega) := L\left(\sup_{N \in \mathbb{N}} \sup_{0 \le t \le T} \left\| X_t^N(\omega) \right\|_V \right)$$

for every $\omega \in \Omega$. Furthermore, we obtain

$$\begin{aligned}
&\left\| X_t^N - X_t^M \right\|_V \\
&\le \left\| \int_0^t P_N\, S_{(t-s)} F(X_s^N)\, ds - \int_0^t P_M\, S_{(t-s)} F(X_s^M)\, ds \right\|_V + \left\| P_N\left(O_t\right) - P_M\left(O_t\right) \right\|_V \\
&\le \left\| \int_0^t P_N\, S_{(t-s)} F(X_s^N)\, ds - \int_0^t P_N\, S_{(t-s)} F(X_s^M)\, ds \right\|_V \\
&\quad + \left\| \int_0^t P_N\, S_{(t-s)} F(X_s^M)\, ds - \int_0^t P_M\, S_{(t-s)} F(X_s^M)\, ds \right\|_V + R \cdot N^{-\gamma} + R \cdot M^{-\gamma}
\end{aligned}$$

and therefore

$$
\begin{aligned}
&\left\|X_t^N - X_t^M\right\|_V \\
&\leq \int_0^t \left\|P_N\, S_{(t-s)}\right\|_{L(W,V)} \cdot \left\|F(X_s^N) - F(X_s^M)\right\|_W \, ds \\
&\quad + \int_0^t \left\|P_N\, S_{(t-s)} - P_M\, S_{(t-s)}\right\|_{L(W,V)} \cdot \left\|F(X_s^M)\right\|_W \, ds + R \cdot N^{-\gamma} + R \cdot M^{-\gamma} \\
&\leq \int_0^t R \cdot (t-s)^{-\alpha} \cdot \left\|F(X_s^N) - F(X_s^M)\right\|_W \, ds \\
&\quad + \int_0^t \left\|P_N\, S_{(t-s)} - P_M\, S_{(t-s)}\right\|_{L(W,V)} \cdot R \, ds + R \cdot N^{-\gamma} + R \cdot M^{-\gamma}
\end{aligned}
$$

for every $N, M \in \mathbb{N}$ and every $t \in [0,T]$. Hence, we have

$$
\begin{aligned}
&\left\|X_t^N - X_t^M\right\|_V \\
&\leq R \int_0^t (t-s)^{-\alpha} \left\|F\left(X_s^N\right) - F\left(X_s^M\right)\right\|_W \, ds + R \cdot N^{-\gamma} + R \cdot M^{-\gamma} \\
&\quad + R \int_0^t \left\|P_N S_{(t-s)} - S_{(t-s)}\right\|_{L(W,V)} + \left\|S_{(t-s)} - P_M S_{(t-s)}\right\|_{L(W,V)} \, ds \\
&\leq R\, Z \int_0^t (t-s)^{-\alpha} \left\|X_s^N - X_s^M\right\|_V \, ds + R \cdot N^{-\gamma} + R \cdot M^{-\gamma} \\
&\quad + R \int_0^t R\left(N^{-\gamma} + M^{-\gamma}\right)(t-s)^{-\alpha} \, ds
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|X_t^N - X_t^M\right\|_V &\leq R\, Z \int_0^t (t-s)^{-\alpha} \left\|X_s^N - X_s^M\right\|_V \, ds + R\left(N^{-\gamma} + M^{-\gamma}\right) \\
&\quad + R^2(N^{-\gamma} + M^{-\gamma}) \int_0^T s^{-\alpha} \, ds \\
&\leq R\, Z \int_0^t (t-s)^{-\alpha} \left\|X_s^N - X_s^M\right\|_V \, ds + R(N^{-\gamma} + M^{-\gamma}) \\
&\quad + R^2(N^{-\gamma} + M^{-\gamma}) \frac{T^{(1-\alpha)}}{(1-\alpha)} \\
&\leq R\, Z \int_0^t (t-s)^{-\alpha} \left\|X_s^N - X_s^M\right\|_V \, ds + (R + R^4)(N^{-\gamma} + M^{-\gamma})
\end{aligned}
$$

for every $N, M \in \mathbb{N}$ and every $t \in [0,T]$. Therefore, Lemma 5.10 yields

$$
\begin{aligned}
\left\|X_t^N - X_t^M\right\|_V &\leq \mathrm{E}_{(1-\alpha)}\left(t\,(R\,Z\,\Gamma(1-\alpha))^{\frac{1}{(1-\alpha)}}\right)\left(R + R^4\right)\left(N^{-\gamma} + M^{-\gamma}\right) \\
&\leq \mathrm{E}_{(1-\alpha)}\left(T\,(R\,Z\,\Gamma(1-\alpha))^{\frac{1}{(1-\alpha)}}\right)\left(2R^4\right)\left(N^{-\gamma} + M^{-\gamma}\right) \qquad (3.2)
\end{aligned}
$$

for every $N, M \in \mathbb{N}$ and every $t \in [0,T]$. This shows that $\left(X^N(\omega)\right)_{N \in \mathbb{N}}$ is a Cauchy-sequence in $C([0,T], V)$ for every $\omega \in \Omega$. Since $C([0,T], V)$ is complete, we can define the stochastic process $X : [0,T] \times \Omega \to V$ with continuous sample paths by

$X_t(\omega) := \lim_{N\to\infty} X_t^N(\omega)$ for every $t \in [0,T]$ and every $\omega \in \Omega$. Hence, we obtain

$$\begin{aligned} X_t(\omega) &= \lim_{N\to\infty} \left(X_t^N(\omega)\right) = \lim_{N\to\infty} \left(\int_0^t P_N\, S_{(t-s)} F(X_s^N(\omega))\, ds + P_N\left(O_t(\omega)\right) \right) \\ &= \lim_{N\to\infty} \left(\int_0^t P_N\, S_{(t-s)} F(X_s^N(\omega))\, ds \right) + O_t(\omega) = \int_0^t S_{(t-s)} F(X_s(\omega))\, ds + O_t(\omega) \end{aligned}$$

for every $t \in [0,T]$ and every $\omega \in \Omega$. If $Y : [0,T] \times \Omega \to V$ is a further stochastic process with continuous sample paths, which fulfills

$$Y_t(\omega) = \int_0^t S_{(t-s)} F(Y_s(\omega))\, ds + O_t(\omega)$$

for every $t \in [0,T]$ and every $\omega \in \Omega$, then we obtain

$$\begin{aligned} \|X_t - Y_t\|_V &= \left\| \int_0^t S_{(t-s)} \left(F(X_s) - F(Y_s)\right) ds \right\|_V \\ &\le \int_0^t \left\| S_{(t-s)} \left(F(X_s) - F(Y_s)\right) \right\|_V\, ds \\ &\le \int_0^t \|S_{(t-s)}\|_{L(W,V)} \cdot \|F(X_s) - F(Y_s)\|_W\, ds \end{aligned}$$

for every $t \in [0,T]$. Hence, we have

$$\begin{aligned} \|X_t - Y_t\|_V &\le \int_0^t (t-s)^{-\alpha} \cdot \|F(X_s) - F(Y_s)\|_W\, ds \\ &\le \int_0^t (t-s)^{-\alpha} \cdot L\left(\sup_{0\le r\le T} \|X_r\|_V + \sup_{0\le r\le T} \|Y_r\|_V \right) \cdot \|X_s - Y_s\|_V\, ds \\ &= L\left(\sup_{0\le r\le T} \|X_r\|_V + \sup_{0\le r\le T} \|Y_r\|_V \right) \cdot \int_0^t (t-s)^{-\alpha} \|X_s - Y_s\|_V\, ds \end{aligned}$$

for every $t \in [0,T]$. Due to Lemma 5.10, we obtain $X_t(\omega) = Y_t(\omega)$ for every $t \in [0,T]$, $\omega \in \Omega$, which shows that $X : [0,T] \times \Omega \to V$ is the pathwise unique stochastic process with continuous sample paths satisfying equation (3.1). Moreover, inequality (3.2) yields

$$\|X_t^N - X_t\|_V \le \mathrm{E}_{(1-\alpha)}\left(T\left(R\, Z\, \Gamma(1-\alpha)\right)^{\frac{1}{(1-\alpha)}} \right) \cdot 2R^4 \cdot N^{-\gamma}$$

for every $t \in [0,T]$, $N \in \mathbb{N}$ and therefore

$$\sup_{0\le t\le T} \|X_t - X_t^N\|_V \le C \cdot N^{-\gamma}$$

for every $N \in \mathbb{N}$, where the $\mathcal{F}/\mathcal{B}[0,\infty))$-measurable mapping $C : \Omega \to [0,\infty)$ is given by

$$C(\omega) := 2 \cdot (R(\omega))^4 \cdot \mathrm{E}_{(1-\alpha)}\left(T\left(R(\omega) Z(\omega) \Gamma(1-\alpha)\right)^{\frac{1}{(1-\alpha)}} \right)$$

for every $\omega \in \Omega$. □

4. Examples. In this section some examples of the setting in Section 2 are presented.

4.1. Stochastic heat equation. In this subsection an important example of Assumption 3 is presented. We consider a linear equation with $F = 0$ and thus consider only the approximation of the Ornstein-Uhlenbeck process O.

To this end let $d \in \{1, 2, 3\}$ and let $V = W = C([0, 1]^d, \mathbb{R})$ be the $\mathbb{R}$-Banach space of continuous functions from $[0, 1]^d$ to $\mathbb{R}$ equipped with the norm

$$\|v\|_V = \|v\|_W = \|v\|_{C([0,1]^d,\mathbb{R})} := \sup_{x\in[0,1]^d} |v(x)|$$

for every $v \in V = W$, where $|\cdot|$ is the absolute value of a real number. Moreover, consider the continuous functions

$$e_i : [0, 1]^d \to \mathbb{R}, \qquad e_i(x) = 2^{\frac{d}{2}} \sin(i_1\pi x_1) \ldots \sin(i_d\pi x_d), \qquad x \in [0, 1]^d \tag{4.1}$$

and the real numbers

$$\lambda_i = \pi^2(i_1^2 + \ldots + i_d^2) \in \mathbb{R} \tag{4.2}$$

for every $i = (i_1, \ldots, i_d) \in \mathbb{N}^d$. Additionally, suppose that the bounded linear operators $P_N : C([0, 1]^d, \mathbb{R}) \to C([0, 1]^d, \mathbb{R})$ are given by

$$(P_N(v))(x) = \sum_{i\in\{1,\ldots,N\}^d} \int_{(0,1)^d} e_i(s)\, v(s)\, ds\, e_i(x) \tag{4.3}$$

for every $x \in [0, 1]^d$, $v \in C([0, 1]^d, \mathbb{R})$ and every $N \in \mathbb{N}$. Before we present the stochastic process satisfying Assumption 3, we consider the following example of Assumption 1.

Lemma 4.1. *Let $d \in \{1, 2, 3\}$. Then, the mapping $S : (0, T] \to L\left(C\left([0, 1]^d, \mathbb{R}\right)\right)$ given by*

$$(S_t v)(x) = \sum_{i\in\mathbb{N}^d} e^{-\lambda_i t} \int_{(0,1)^d} e_i(s)\, v(s)\, ds \cdot e_i(x)$$

for every $t \in (0, T]$, $x \in [0, 1]^d$ and every $v \in C\left([0, 1]^d, \mathbb{R}\right)$ satisfies Assumption 1 for every $\gamma \in (0, 2 - \frac{d}{2})$. Here, the functions $e_i \in C\left([0, 1]^d, \mathbb{R}\right)$, $i \in \mathbb{N}^d$, and the real numbers $\lambda_i \in \mathbb{R}$, $i \in \mathbb{N}^d$, are given in (4.1) and (4.2).

Of course this is simply the semigroup generated by the Laplacian with Dirichlet boundary conditions (see e.g. Section 3.8.1 in [30]). Other boundary conditions such as Neumann or periodic boundary conditions could also be considered here. We now present the promised example of Assumption 3. We consider a stochastic convolution of the semigroup S constructed in Lemma 4.1 and a cylindrical Wiener process. The following result provides an appropriate version of such a process, in which the initial value of the stochastic evolution equation (3.1) is additionally incorporated.

Proposition 4.2. *Let $d \in \{1, 2, 3\}$, let $V = C\left([0, 1]^d, \mathbb{R}\right)$, let $\rho > 0$, let $\beta^i : [0, T] \times \Omega \to \mathbb{R}$, $i \in \mathbb{N}^d$, be a family of independent standard Brownian motions with continuous sample paths and let $b : \mathbb{N}^d \to \mathbb{R}$ be a given function such that $\sum_{i\in\mathbb{N}^d} \left(i_1^2 + \ldots + i_d^2\right)^{(\rho-1)} |b(i)|^2 < \infty$. Furthermore, suppose that $\xi : \Omega \to V$ is a $\mathcal{F}/\mathcal{B}(V)$-measurable mapping with $\sup_{N\in\mathbb{N}} \left(N^\rho \left\|\xi(\omega) - P_N\left(\xi(\omega)\right)\right\|_V\right) < \infty$ for every*

$\omega \in \Omega$. Then, there exists a stochastic process $O : [0,T] \times \Omega \to V$ with continuous sample paths, which satisfies

$$\mathbb{P}\left[\lim_{N\to\infty} \sup_{0<t\leq T} \left\| O_t - S_t\,\xi - \sum_{i\in\{1,\ldots,N\}^d} b(i) \left(-\lambda_i \int_0^t e^{-\lambda_i(t-s)} \beta_s^i \, ds + \beta_t^i \right) e_i \right\|_V = 0 \right] = 1 \tag{4.4}$$

and

$$\sup_{N\in\mathbb{N}} \sup_{0\leq t\leq T} \left(N^\gamma \left\| O_t(\omega) - P_N\left(O_t(\omega)\right) \right\|_V \right) < \infty$$

for every $\omega \in \Omega$ and every $\gamma \in (0,\rho)$. In particular O satisfies Assumption 3 for every $\gamma \in (0,\rho)$. Here, the functions $e_i \in V$, $i \in \mathbb{N}^d$, the real numbers λ_i, $i \in \mathbb{N}^d$, and the linear operators $P_N : V \to V$, $N \in \mathbb{N}$, are given in (4.1), (4.2) and (4.3).

We remark that the stochastic process constructed in Proposition 4.2 is unique up to indistinguishability. More precisely, if $O : [0,T] \times \Omega \to C\left([0,1]^d, \mathbb{R}\right)$ and $\tilde{O} : [0,T] \times \Omega \to C\left([0,1]^d, \mathbb{R}\right)$ are two stochastic processes with continuous sample paths that satisfy (4.4), then $\mathbb{P}\left[\, O_t = \tilde{O}_t \ \forall\, t \in [0,T] \right] = 1$. Moreover, in the sense of Proposition 4.2 we have

$$O_t = S_t\,\xi + \sum_{i\in\mathbb{N}^d} b(i) \int_0^t e^{-\lambda_i(t-s)} d\beta_s^i \cdot e_i \qquad \mathbb{P}-\text{a.s.}$$

for every $t \in (0,T]$. In that sense O includes the initial value and a stochastic convolution of the semigroup generated by the Laplacian with Dirichlet boundary conditions and a cylindrical Wiener process as it is usually considered in the literature (see e.g. Section 5 in [6]). Note that $O : [0,T] \times \Omega \to C\left([0,1]^d, \mathbb{R}\right)$ is nothing else than the solution of the linear SPDE

$$dO_t = \left[\Delta O_t\right] dt + B\, dW_t, \qquad O_t|_{\partial(0,1)^d} \equiv 0, \qquad O_0 = \xi$$

for $t \in [0,T]$ in $C\left([0,1]^d, \mathbb{R}\right)$, where W_t, $t \in [0,T]$, is a cylindrical I-Wiener process on $L^2\left((0,1)^d, \mathbb{R}\right)$ and where $B : L^2((0,1)^d, \mathbb{R}) \to L^2((0,1)^d, \mathbb{R})$ is given by

$$Bv = \sum_{i\in\mathbb{N}^d} b(i) \int_{(0,1)^d} e_i(s)\, v(s)\, ds \cdot e_i \tag{4.5}$$

for every $v \in L^2\left((0,1)^d, \mathbb{R}\right)$. Here, $b : \mathbb{N}^d \to \mathbb{R}$ is the function used in Proposition 4.2 and $L^2\left((0,1)^d, \mathbb{R}\right)$ is the $\mathbb{R}$-Hilbert space of equivalence classes of $\mathcal{B}\left((0,1)^d\right)/\mathcal{B}(\mathbb{R})$-measurable and square integral functions from $(0,1)^d$ to $\mathbb{R}$.

Numerical example. To illustrate Proposition 4.2 we consider the following simple example. Let $d = 2$, $T = 1$, $(\xi(\omega))(x) = 0$ for all $x \in [0,1]^2$, $\omega \in \Omega$ and let $b : \mathbb{N}^2 \to \mathbb{R}$ be given by $b\left((i_1,i_2)\right) = \frac{1}{(i_1+i_2)}$ for all $i = (i_1,i_2) \in \mathbb{N}^2$. In view of Proposition 4.2 we obtain

$$\sup_{0\leq t\leq 1} \sup_{x\in[0,1]^2} \left| O_t(\omega,x) - P_N\left(O_t(\omega,x)\right) \right| \leq C_\gamma(\omega) \cdot N^{-\gamma}$$

for all $\omega \in \Omega$ and all $N \in \mathbb{N}$ with $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mappings $C_\gamma : \Omega \to [0,\infty)$ for every $\gamma \in (0,1)$. Hence, $P_N\left(O_t(\omega,x)\right)$ converges to $O_t(\omega,x)$ uniformly in $t \in [0,1]$

and $x \in [0,1]^2$ with the rate $1-$ as N goes to infinity for all $\omega \in \Omega$. This is illustrated in Figure 4.1, where the expression

$$\sup_{0 \le t \le 1} \sup_{x \in [0,1]^2} \left| O_t(\omega, x) - P_N \left(O_t(\omega, x) \right) \right| \tag{4.6}$$

versus N for $N = 4, 8, 16, \dots, 256$ and two $\omega \in \Omega$ is plotted. Moreover, in Figure 4.2 we plot $O_t(\omega, x)$, $x \in [0,1]^2$, for $t \in \left\{ \frac{1}{1000}, 1 \right\}$ and one random $\omega \in \Omega$.

Finally, note that Proposition 4.2 immediately follows from the following lemma, which is also of independent interest.

LEMMA 4.3. *Let $d \in \mathbb{N}$, let $V = C\left([0,1]^d, \mathbb{R}\right)$, let $\rho > 0$, let $\beta^i : [0,T] \times \Omega \to \mathbb{R}$, $i \in \mathbb{N}^d$, be a family of independent standard Brownian motions with continuous sample paths and let $b : \mathbb{N}^d \to \mathbb{R}$ be a given function such that $\sum_{i \in \mathbb{N}^d} \left(i_1^2 + \ldots + i_d^2 \right)^{(\rho - 1)} |b(i)|^2 < \infty$. Then, there exists a stochastic process $O : [0,T] \times \Omega \to V$, which satisfies*

$$\sup_{0 \le t_1 < t_2 \le T} \frac{\left\| O_{t_2}(\omega) - O_{t_1}(\omega) \right\|_V}{\left(t_2 - t_1\right)^\theta} < \infty, \qquad \sup_{N \in \mathbb{N}} \sup_{0 \le t \le T} \left(\left\| O_t(\omega) - P_N O_t(\omega) \right\|_V N^\gamma \right) < \infty$$

for every $\omega \in \Omega$, every $\theta \in \left(0, \min\left(\frac{1}{2}, \frac{\rho}{2}\right)\right)$, every $\gamma \in (0, \rho)$ and which satisfies

$$\mathbb{P}\left[\lim_{N \to \infty} \sup_{0 \le t \le T} \left\| O_t - \sum_{i \in \{1, \dots, N\}^d} b(i) \left(-\lambda_i \int_0^t e^{-\lambda_i (t-s)} \beta_s^i \, ds + \beta_t^i \right) e_i \right\|_V = 0 \right] = 1, \tag{4.7}$$

$$\sup_{N \in \mathbb{N}} \left[\left(\mathbb{E}\left[\sup_{0 \le t \le T} \left\| O_t - P_N O_t \right\|_V^p \right] \right)^{\frac{1}{p}} N^\gamma \right] + \sup_{0 \le t_1 < t_2 \le T} \frac{\left(\mathbb{E}\left[\left\| O_{t_2} - O_{t_1} \right\|_V^p \right] \right)^{\frac{1}{p}}}{\left(t_2 - t_1\right)^\theta} < \infty$$

for every $p \in [1, \infty)$, every $\theta \in \left(0, \frac{\rho}{2}\right)$, $\theta \le \frac{1}{2}$ and every $\gamma \in (0, \rho)$. Here, the functions $e_i \in V$, $i \in \mathbb{N}^d$, the real numbers λ_i, $i \in \mathbb{N}^d$, and the linear operators $P_N : V \to V$, $N \in \mathbb{N}$, are given in (4.1), (4.2) *and* (4.3).

4.2. Stochastic evolution equations with a globally Lipschitz nonlinearity. If the nonlinearity $F : V \to W$ given in Assumption 2 is globally Lipschitz continuous from V to W, then Assumption 4 is naturally met, which can be seen in the following proposition.

PROPOSITION 4.4. *Suppose that Assumptions 1-3 are fulfilled. If the nonlinearity $F : V \to W$ given in Assumption 2 additionally satisfies $\sup_{v,w \in V, v \ne w} \frac{\|F(v) - F(w)\|_W}{\|v - w\|_V} < \infty$, then Assumption 4 is fulfilled.*

In the remainder of this section we illustrate Theorem 3.1 with a stochastic reaction diffusion equation with a globally Lipschitz nonlinearity. Again we suppose that $V = W = C\left([0,1]^d, \mathbb{R}\right)$ with $d \in \{1, 2, 3\}$ fixed.

LEMMA 4.5. *Let $f : [0,1]^d \times \mathbb{R} \to \mathbb{R}$ be a continuous function, which satisfies*

$$L := \sup_{x \in [0,1]^d} \sup_{\substack{y_1, y_2 \in \mathbb{R} \\ y_1 \ne y_2}} \frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} < \infty.$$

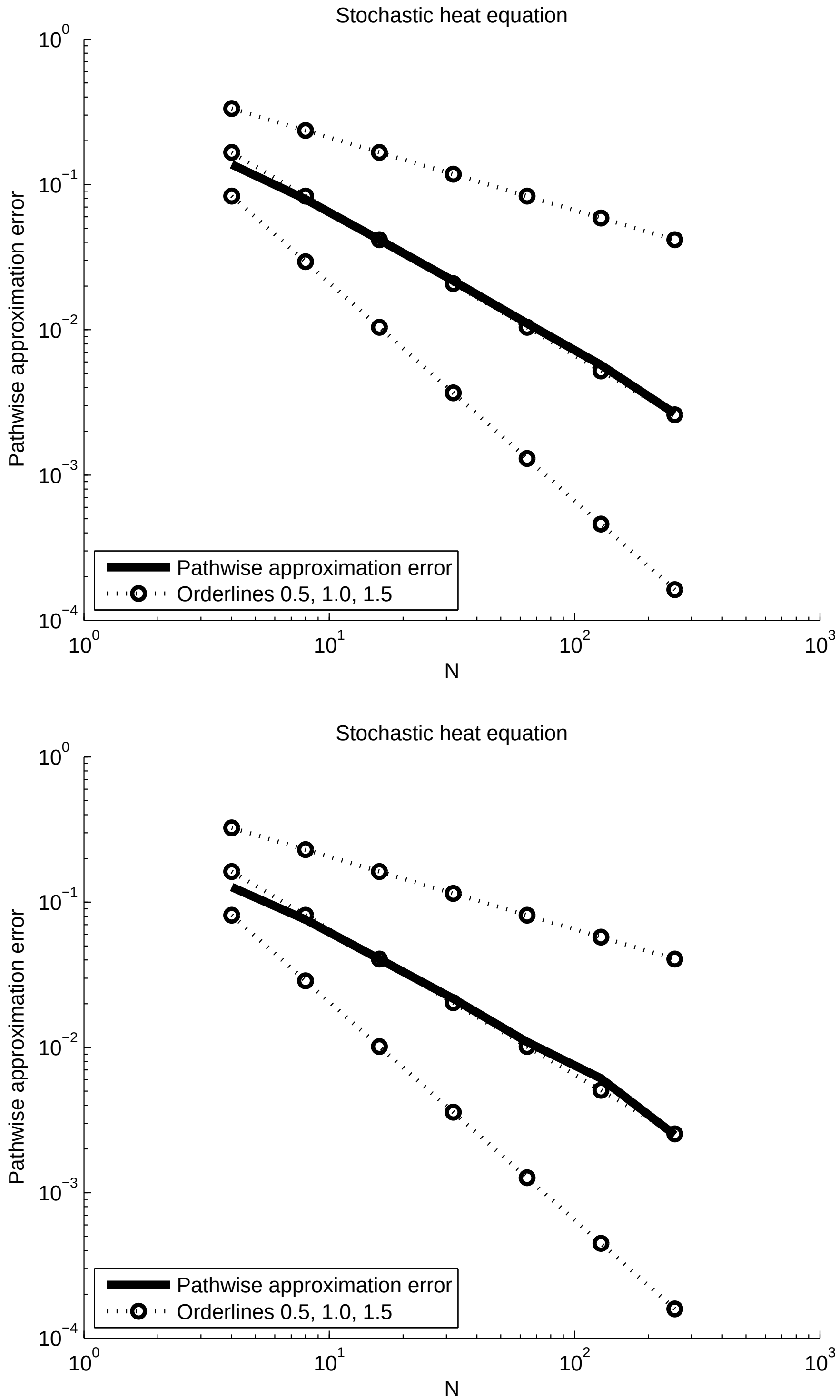


FIG. 4.1. *Pathwise approximation error* (4.6) *versus* N *for* $N = 4, 8, 16, \ldots, 256$ *and two random* $\omega \in \Omega$.

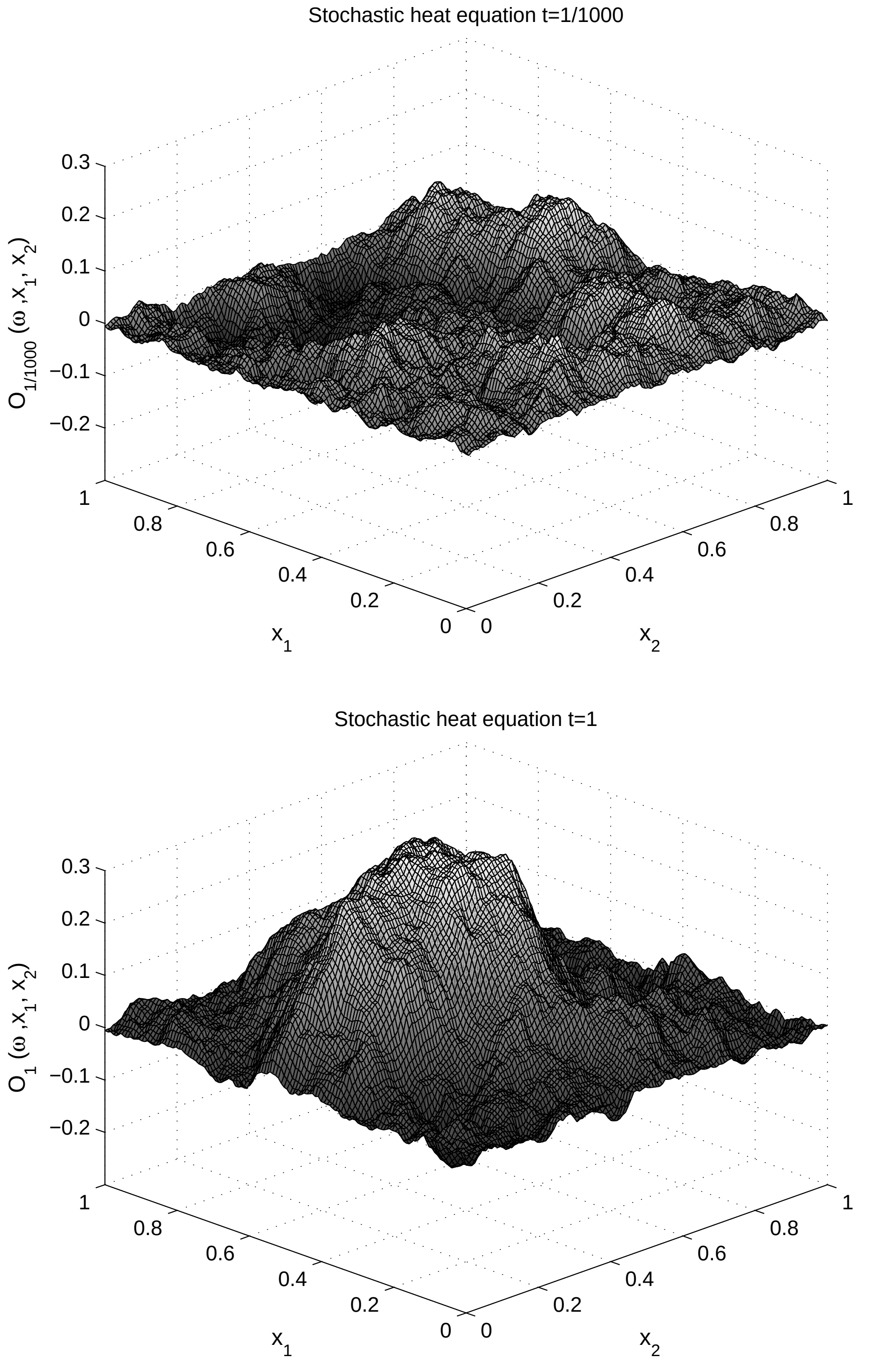


FIG. 4.2. $O_t(\omega, x)$, $x = (x_1, x_2) \in [0,1]^2$, *for* $t \in \left\{\frac{1}{1000}, 1\right\}$ *and one random* $\omega \in \Omega$.

Then, the corresponding Nemytskii operator $F : C([0,1]^d, \mathbb{R}) \to C([0,1]^d, \mathbb{R})$ *given by* $(F(v))(x) = f(x, v(x))$ *for every* $x \in [0,1]^d$ *and every* $v \in C([0,1]^d, \mathbb{R})$ *satisfies* $\sup_{v,w\in V, v\neq w} \frac{\|F(v)-F(w)\|_{C([0,1]^d,\mathbb{R})}}{\|v-w\|_{C([0,1]^d,\mathbb{R})}} < \infty$.

Proof. [Proof of Lemma 4.5] We obtain

$$\begin{aligned}\|F(v) - F(w)\|_{C([0,1]^d,\mathbb{R})} &= \sup_{x\in[0,1]^d} |f(x, v(x)) - f(x, w(x))| \\ &\leq \sup_{x\in[0,1]^d} (L|v(x) - w(x)|) = L\|v - w\|_{C([0,1]^d,\mathbb{R})}\end{aligned}$$

for every $v, w \in C([0,1]^d, \mathbb{R})$ □

Let $P_N : V \to V$, $N \in \mathbb{N}$, $S : (0,T] \to L(V)$, $F : V \to V$ and $O : [0,T] \times \Omega \to V$ be given by (4.3), by Lemma 4.1, Lemma 4.5 and Proposition 4.2. Then, Assumption 4 is fulfilled due to Proposition 4.4 and therefore the assumptions in Theorem 3.1 are fulfilled. The stochastic evolution equation (3.1) reduces in that case to

$$dX_t = [\Delta X_t + f(x, X_t)]\, dt + B\, dW_t, \qquad X_t|_{\partial(0,1)^d} \equiv 0, \qquad X_0 = \xi \tag{4.8}$$

for $t \in [0,T]$ and $x \in [0,1]^d$, where W_t, $t \in [0,T]$, is a cylindrical I-Wiener process on $L^2\left((0,1)^d, \mathbb{R}\right)$, where $\xi : \Omega \to V$ is used in Proposition 4.2 and where $B : L^2((0,1), \mathbb{R}) \to L^2((0,1), \mathbb{R})$ is given by (4.5) with $b : \mathbb{N}^d \to \mathbb{R}$ used in Proposition 4.2. Moreover, the finite dimensional SODEs (2.1) reduces to

$$dX_t^N = \left[\Delta X_t^N + P_N f(x, X_t^N)\right] dt + P_N B\, dW_t, \qquad X_t^N|_{\partial(0,1)^d} \equiv 0, \qquad X_0^N = P_N(\xi)$$

for $t \in [0,T]$ and $x \in [0,1]^d$.

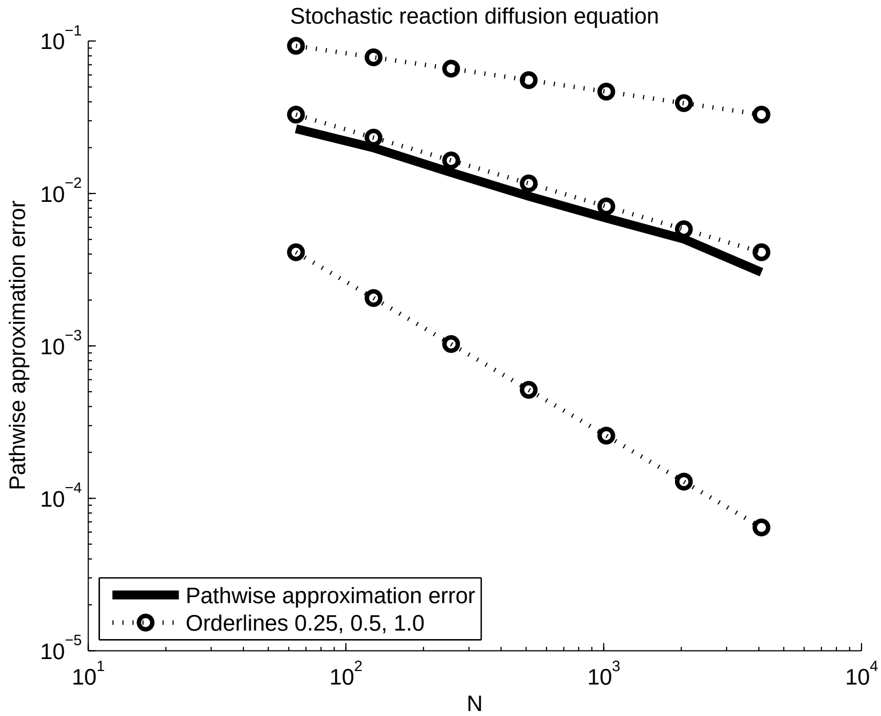
Numerical Example. In order to do numerical computations we consider the following simple example. Let $T = \frac{1}{20}$, $d = 1$, $(\xi(\omega))(x) = \frac{6}{5}\sin(\pi x)$ for all $x \in [0,1]$, $\omega \in \Omega$, let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be given by $f(x,y) = 300\frac{(1-y)}{(1+y^4)}$ for all $x \in [0,1]$, $y \in \mathbb{R}$ and let $b : \mathbb{N} \to \mathbb{R}$ be given by $b(i) = \frac{1}{3}$ for all $i \in \mathbb{N}$. The corresponding SPDE (4.8) then reduces to

$$dX_t = \left[\frac{\partial^2}{\partial x^2} X_t + 300\frac{(1 - X_t)}{(1 + X_t^4)}\right] dt + \tfrac{1}{3} dW_t, \quad X_0(x) = \tfrac{6}{5}\sin(\pi x) \tag{4.9}$$

with $X_t(0) = X_t(1) = 0$ for $t \in [0, \frac{1}{20}]$ and $x \in [0,1]$ on $C([0,1], \mathbb{R})$. In view of Lemma 4.1 and Proposition 4.2, Theorem 3.1 yields the existence of $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mappings $C_\gamma : \Omega \to [0,\infty)$ such that

$$\sup_{0\leq t\leq \frac{1}{20}} \sup_{0\leq x\leq 1} \left| X_t(\omega, x) - X_t^N(\omega, x) \right| \leq C_\gamma(\omega) \cdot N^{-\gamma} \tag{4.10}$$

holds for every $\omega \in \Omega$, $N \in \mathbb{N}$ and every $\gamma \in (0, \frac{1}{2})$. Hence, $X_t^N(\omega, x)$ converges to $X_t(\omega, x)$ uniformly in $t \in [0, \frac{1}{20}]$ and $x \in [0,1]$ with the rate $\frac{1}{2}-$ as N goes to infinity for all $\omega \in \Omega$, which is also illustrated in Figure 4.3. Furthermore, $X_t(\omega, x)$, $x \in [0,1]$, is plotted in Figure 4.5 for $t \in \left\{0, \frac{1}{450}, \frac{1}{200}, \frac{3}{200}, \frac{1}{20}\right\}$ and one $\omega \in \Omega$.



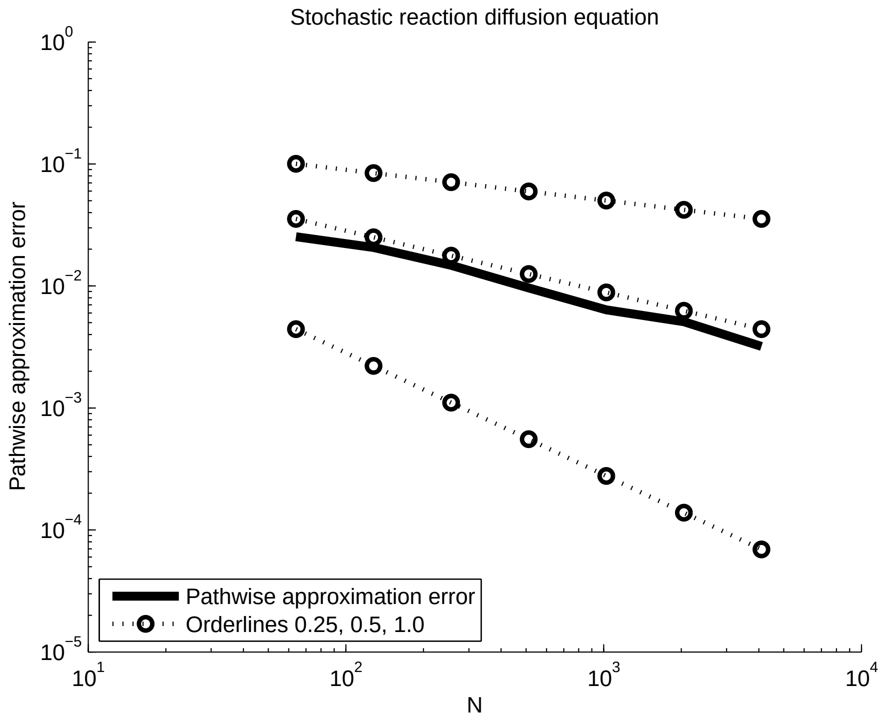


FIG. 4.3. *Pathwise approximation error* (4.10) *versus* N *for* $N = 64, 128, \ldots, 4096$ *and two random* $\omega \in \Omega$.

4.3. Stochastic Burgers equation. In order to formulate the stochastic Burgers equation we denote by $\left(L^2\left((0,1),\mathbb{R}\right),\|\cdot\|_{L^2}\right)$ the $\mathbb{R}$-Hilbert space of equivalence classes of $\mathcal{B}(0,1)/\mathcal{B}(\mathbb{R})$-measurable and square integrable functions from $(0,1)$ to $\mathbb{R}$. The norm and the scalar product in $L^2\left((0,1),\mathbb{R}\right)$ are given by

$$\|v\|_{L^2} := \int_0^1 |v(x)|^2 \, dx, \qquad \langle v, w\rangle_{L^2} := \int_0^1 v(x)\, w(x)\, dx$$

for every $v, w \in L^2\left((0,1),\mathbb{R}\right)$. We also denote by

$$\dot{L}^2\left((0,1),\mathbb{R}\right) := \left\{ v \in L^2\left((0,1),\mathbb{R}\right) \,\middle|\, \int_0^1 v(x)\, dx = 0 \right\}$$

equipped with the norm $\|\cdot\|_{L^2}$ and the scalar product $\langle\cdot,\cdot\rangle_{L^2}$ the $\mathbb{R}$-Hilbert subspace of $L^2\left((0,1),\mathbb{R}\right)$ of equivalence classes of functions with zero mean. Moreover, let $D\left((0,1),\mathbb{R}\right) = C^\infty_{\mathrm{cpt}}\left((0,1),\mathbb{R}\right)$ be the $\mathbb{R}$-vector space of infinitely often differentiable functions with compact support in $(0,1)$ and let $D'\left((0,1),\mathbb{R}\right)$ be the $\mathbb{R}$-vector space of real valued distributions on $(0,1)$. Furthermore, let

$$H^k\left((0,1),\mathbb{R}\right) := \left\{ v \in L^2\left((0,1),\mathbb{R}\right) \,\middle|\, \partial^n v \in L^2\left((0,1),\mathbb{R}\right) \ \forall\, n \in \{0,1,\ldots,k\} \right\}$$

with the norm and the scalar product given by

$$\|v\|_{H^k} := \left(\sum_{n=0}^{k} \left(\|\partial^n v\|_{L^2} \right)^2 \right)^{\frac{1}{2}}, \qquad \langle v, w\rangle_{H^k} := \sum_{n=0}^{k} \langle \partial^n v, \partial^n w\rangle_{L^2}$$

for every $v, w \in H^k\left((0,1),\mathbb{R}\right)$ be the $\mathbb{R}$-Hilbert space of k-times weakly differentiable functions in $L^2\left((0,1),\mathbb{R}\right)$ for every $k \in \mathbb{N}$. Additionally, we denote by

$$H^k_0\left((0,1),\mathbb{R}\right) := \overline{D\left((0,1),\mathbb{R}\right)}^{H^k}$$

the closure of $D\left((0,1),\mathbb{R}\right)$ in the $\mathbb{R}$-Hilbert space $\left(H^k\left((0,1),\mathbb{R}\right),\langle\cdot,\cdot\rangle_{H^k}\right)$ for every $k \in \mathbb{N}$. We use the norm and the scalar product

$$\|v\|_{H^k_0} := \left\|\partial^k v\right\|_{L^2}, \qquad \langle v, w\rangle_{H^k_0} := \left\langle \partial^k v, \partial^k w\right\rangle_{L^2}$$

for every $v, w \in H^k_0\left((0,1),\mathbb{R}\right)$ in $H^k_0\left((0,1),\mathbb{R}\right)$ for every $k \in \mathbb{N}$. Due to Poincaré's inequality $\|\cdot\|_{H^k_0}$ is equivalent to $\|\cdot\|_{H^k}$ on $H^k_0\left((0,1),\mathbb{R}\right)$ for every $k \in \mathbb{N}$ (see Proposition 5.8 in [28]). Finally, we denote by

$$H^{-k}\left((0,1),\mathbb{R}\right) := \left(H^k_0\left((0,1),\mathbb{R}\right), \|\cdot\|_{H^k_0} \right)' \subset D'\left((0,1),\mathbb{R}\right)$$

the topological Dual space of $H^k_0\left((0,1),\mathbb{R}\right)$ for every $k \in \mathbb{N}$. Due to the embedding

$$L^2\left((0,1),\mathbb{R}\right) \to D'\left((0,1),\mathbb{R}\right), \qquad v \mapsto \langle v, \cdot\rangle_{L^2}$$

for every $v \in L^2\left((0,1),\mathbb{R}\right)$, we obtain

$$H^k\left((0,1),\mathbb{R}\right) \subset H^n\left((0,1),\mathbb{R}\right) \subset L^2\left((0,1),\mathbb{R}\right) \subset H^{-n}\left((0,1),\mathbb{R}\right) \subset H^{-k}\left((0,1),\mathbb{R}\right)$$

for every $n, k \in \mathbb{N}$ with $n \leq k$. We also refer to Chapter 5 in [28] for a more detailed consideration of these and many more Sobolev spaces.

In view of this scaling of spaces let $W = H^{-1}((0,1), \mathbb{R})$ and let $V = C([0,1], \mathbb{R})$ be the $\mathbb{R}$-Banach space of continuous functions from $[0,1]$ to $\mathbb{R}$. As in Section 4.1 and Section 4.2 we use the projection operators $P_N : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ given by

$$(P_N(v))(x) = \sum_{n=1}^{N} 2 \left(\int_0^1 \sin(n\pi s)\, v(s)\, ds \right) \sin(n\pi x) \tag{4.11}$$

for every $x \in [0,1]$, $v \in C([0,1], \mathbb{R})$ and every $N \in \mathbb{N}$. The semigroup is constructed in the following lemma here.

Lemma 4.6. *The mapping $S : (0,T] \to L\big(H^{-1}((0,1), \mathbb{R}), C([0,1], \mathbb{R})\big)$ given by*

$$(S_t(w))(x) = \sum_{n=1}^{\infty} \left(2 \cdot e^{-n^2\pi^2 t} \cdot w(\sin(n\pi(\cdot))) \cdot \sin(n\pi x) \right)$$

for every $x \in [0,1]$, $w \in H^{-1}((0,1), \mathbb{R})$ and every $t \in (0,T]$ is well defined and satisfies Assumption 1 for every $\gamma \in (0, \frac{1}{2})$.

In order to describe the nonlinearity of the stochastic Burgers equation, we use the following fact concerning distributional derivatives in $L^2((0,1), \mathbb{R})$.

Lemma 4.7. *The mapping $\partial : L^2((0,1), \mathbb{R}) \to H^{-1}(0,1), \mathbb{R})$ given by*

$$(\partial v)(\varphi) = (v')(\varphi) := -\langle v, \varphi' \rangle_{L^2} = -\int_0^1 v(x)\, \varphi'(x)\, dx$$

for every $\varphi \in H^1_0((0,1), \mathbb{R})$ and every $v \in L^2((0,1), \mathbb{R})$ is a surjective bounded linear mapping from $L^2((0,1), \mathbb{R})$ to $H^{-1}((0,1), \mathbb{R})$ with $\|\partial v\|_{H^{-1}} \leq \|v\|_{L^2}$ for every $v \in L^2((0,1), \mathbb{R})$. Additionally, we have $\partial \left(\dot{L}^2((0,1), \mathbb{R}) \right) = H^{-1}((0,1), \mathbb{R})$ and $\|\partial v\|_{H^{-1}} = \|v\|_{L^2}$ for every $v \in \dot{L}^2((0,1), \mathbb{R})$. Finally, it holds

$$\|w\|_{H^{-1}} = \sum_{n=1}^{\infty} \frac{\left| w\left(\sqrt{2} \sin(n\pi(\cdot))\right) \right|^2}{n^2 \pi^2}$$

for every $w \in H^{-1}((0,1), \mathbb{R})$.

The nonlinearity is then given in the following lemma.

Lemma 4.8. *Let $c \in \mathbb{R}$ be a fixed real number. Then, the mapping*

$$F : C([0,1], \mathbb{R}) \to H^{-1}((0,1), \mathbb{R}), \qquad F(v) = c\, \partial\left(v^2\right)$$

for every $v \in C([0,1], \mathbb{R})$ satisfies Assumption 2.

Proof. [Proof of Lemma 4.8] We have

$$\|F(v) - F(w)\|_{H^{-1}} = \|c\, \partial(v^2) - c\, \partial(w^2)\|_{H^{-1}} \leq |c| \cdot \|v^2 - w^2\|_{L^2}$$

and therefore

$$\begin{aligned} \|F(v) - F(w)\|_{H^{-1}} &\leq |c| \cdot \|(v+w) \cdot (v-w)\|_{C([0,1],\mathbb{R})} \\ &\leq |c| \cdot (\|v\|_{C([0,1],\mathbb{R})} + \|w\|_{C([0,1],\mathbb{R})}) \cdot \|v - w\|_{C([0,1],\mathbb{R})} \end{aligned}$$

for every $v, w \in C([0,1],\mathbb{R})$. This yields

$$\|F(v) - F(w)\|_{H^{-1}} \leq (2r|c|)\, \|v - w\|_{C([0,1],\mathbb{R})}$$

for every $v, w \in C([0,1],\mathbb{R})$ with $\|v\|_{C([0,1],\mathbb{R})}, \|w\|_{C([0,1],\mathbb{R})} \leq r$ and every $r > 0$. □

With this type of nonlinearities Assumption 4 is fulfilled, which can be seen in the following lemma.

LEMMA 4.9. *Let $V = C([0,1],\mathbb{R})$, $W = H^{-1}((0,1),\mathbb{R})$ and let $S : (0,T] \to L(W,V)$, $F : V \to W$ and $P_N : V \to V$, $N \in \mathbb{N}$, be given by Lemma 4.6, Lemma 4.8 and* (4.11)*. Moreover, let $O : [0,T] \times \Omega \to V$ be a stochastic process with continuous sample paths and with* $\sup_{N\in\mathbb{N}} \sup_{0\leq t\leq T} \|P_N(O_t(\omega))\|_V < \infty$ *for every $\omega \in \Omega$. Then, Assumption 4 is fulfilled.*

Numerical Example. Finally, we consider the stochastic evolution equation (3.1) with $S : (0,T] \to L(W,V)$, $F : V \to W$ and $O : [0,T] \times \Omega \to V$ given by Lemma 4.6, Lemma 4.8 and Proposition 4.2 with the parameters $c = -30$, $T = \frac{1}{20}$, $\xi(\omega) = \frac{6}{5}\sin(\pi x)$ for every $\omega \in \Omega$ and $b(i) = \frac{1}{3}$ for every $i \in \mathbb{N}$. The stochastic evolution equation (3.1) then reduces to

$$dX_t = \left[\frac{\partial^2}{\partial x^2} X_t - 60 \cdot X_t \frac{\partial}{\partial x} X_t\right] dt + \frac{1}{3} dW_t, \qquad X_0(x) = \frac{6}{5}\sin(\pi x) \tag{4.12}$$

with $X_t(0) = X_t(1) = 0$ for $t \in [0, \frac{1}{20}]$ and $x \in [0,1]$, while the finite dimensional SODEs (2.1) reduce to

$$dX_t^N = \left[\frac{\partial^2}{\partial x^2} X_t^N - 60 \cdot P_N\left(X_t^N \frac{\partial}{\partial x} X_t^N\right)\right] dt + \frac{1}{3} P_N\, dW_t, \qquad X_0^N(x) = \frac{6}{5}\sin(\pi x) \tag{4.13}$$

with $X_t^N(0) = X_t^N(1) = 0$ for $t \in [0, \frac{1}{20}]$, $x \in [0,1]$ and all $N \in \mathbb{N}$. Here, W_t, $t \in [0, \frac{1}{20}]$, is a cylindrical I-Wiener process on $L^2((0,1),\mathbb{R})$. Theorem 3.1 yields then the existence of a unique solution $X : [0, \frac{1}{20}] \times \Omega \to C([0,1],\mathbb{R})$ of the SPDE (4.12) and also the estimate

$$\sup_{0\leq t\leq \frac{1}{20}} \sup_{0\leq x\leq 1} \left| X_t(\omega, x) - X_t^N(\omega, x) \right| \leq C_\gamma(\omega) \cdot N^{-\gamma} \tag{4.14}$$

for every $N \in \mathbb{N}$ and every $\omega \in \Omega$ with appropriate $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mappings $C_\gamma : \Omega \to [0,\infty)$ for every $\gamma \in (0, \frac{1}{2})$. Hence, the solutions $X_t^N(\omega, x)$ of the finite dimensional SODEs (4.13) converge uniformly in $t \in [0, \frac{1}{20}]$ and $x \in [0,1]$ to the solution $X_t(\omega, x)$ of the stochastic Burgers equation (4.12) with the rate $\frac{1}{2}-$ as N goes to infinity for all $\omega \in \Omega$ (see (4.14)). This convergence rate seems to be sharp, which can be seen in Figure 4.4. Finally, in Figure 4.5, $X_t(\omega, x)$, $x \in [0,1]$, is plotted for $t \in \left\{0, \frac{1}{450}, \frac{1}{200}, \frac{3}{200}, \frac{1}{20}\right\}$ and one $\omega \in \Omega$.

5. Proofs.

In this section we collect all technical proofs of the previous sections.

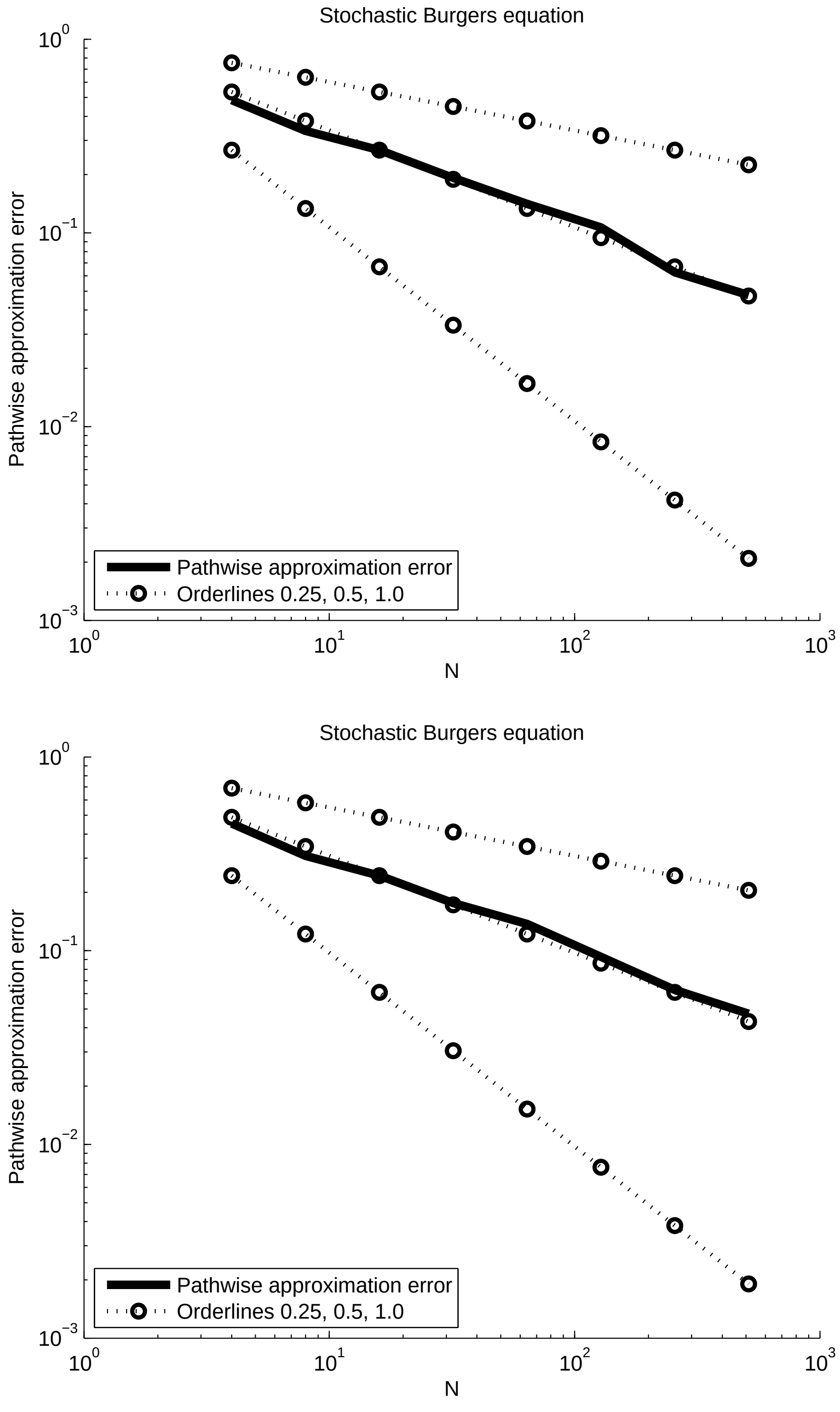


FIG. 4.4. *Pathwise approximation error* (4.14) *versus* N *for* $N = 32, 64, \ldots, 512$ *and two random* $\omega \in \Omega$.

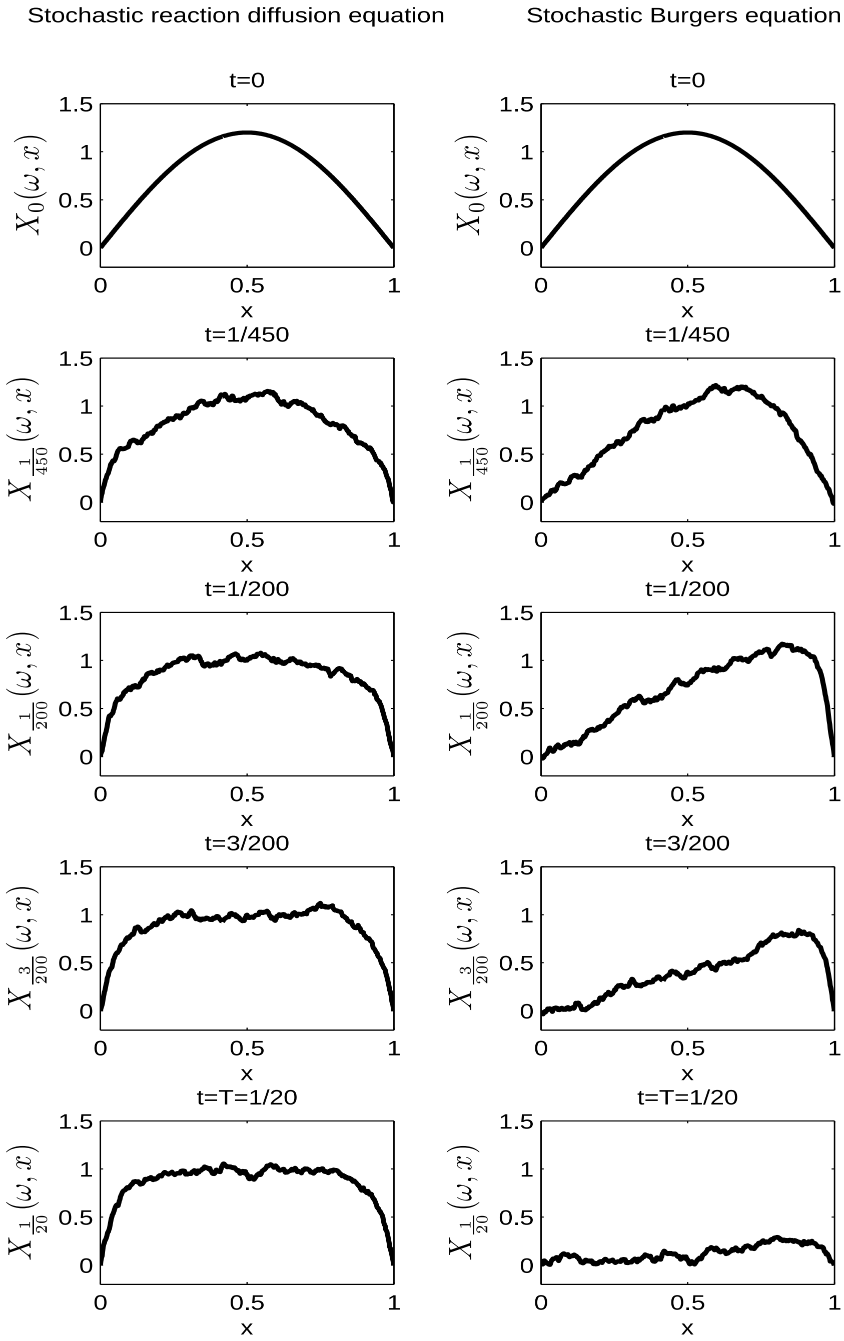


FIG. 4.5. *Stochastic reaction diffusion equation $X_t(\omega, x)$, $x \in [0, 1]$, given by* (4.9) *and stochastic Burgers equation $X_t(\omega, x)$, $x \in [0, 1]$, given by* (4.12) *for $t \in \left\{0, \frac{1}{450}, \frac{1}{200}, \frac{3}{200}, \frac{1}{20}\right\}$ and one random $\omega \in \Omega$.*

5.1. Proof of Proposition 4.4. *Proof.* [Proof of Proposition 4.4] Throughout this proof we use the $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mapping $C:\Omega\to[0,\infty)$ given by

$$C(\omega) := 1 + \sup_{N\in\mathbb{N}}\sup_{0<t\leq T}\Big(t^{\alpha}\left\|P_N S_t\right\|_{L(W,V)}\Big) + \sup_{\substack{v,w\in V\\ v\neq w}}\frac{\|F(v)-F(w)\|_W}{\|v-w\|_V} + \sup_{N\in\mathbb{N}}\sup_{0\leq t\leq T}\|P_N\left(O_t(\omega)\right)\|_V$$

for every $\omega\in\Omega$, which is finite due to Assumptions 1-3 and since we assumed F to be globally Lipschitz continuous. Moreover, let $C\left([0,T],V\right)$ be the $\mathbb{R}$-vector space of continuous functions from $[0,T]$ to V and let

$$\mathcal{S} := \left\{Y:[0,T]\times\Omega\to V\,\middle|\, Y \text{ stochastic process with continuous sample paths}\right\}$$

be the $\mathbb{R}$-vector space of stochastic processes with continuous sample paths. Of course, we have $Y(\omega)\in C\left([0,T],V\right)$ for every $\omega\in\Omega$ and every $Y\in\mathcal{S}$. We equip the space $C\left([0,T],V\right)$ with the norms $\|y\|_{\mu} := \sup_{0\leq t\leq T}\left(e^{\mu t}\left\|y(t)\right\|_V\right)$ for $\mu\in\mathbb{R}$. Hence, $\Big(C\left([0,T],V\right), \|\cdot\|_{\mu}\Big)$ is a $\mathbb{R}$-Banach space for every $\mu\in\mathbb{R}$. Moreover, we define the mappings $\Phi^N_\omega : C\left([0,T],V\right)\to C\left([0,T],V\right)$ by

$$(\Phi^N_\omega y)(t) = \int_0^t P_N\, S_{(t-s)}\, F\Big(y(s) + P_N O_s(\omega)\Big)\, ds$$

for every $t\in[0,T]$, $y\in C([0,T],V)$, $\omega\in\Omega$ and every $N\in\mathbb{N}$. Note that

$$\begin{aligned}
&\|(\Phi^N_\omega y)(t_2) - (\Phi^N_\omega y)(t_1)\|_V\\
&\leq \int_{t_1}^{t_2}\|P_N\, S_{(t_2-s)}\|_{L(W,V)}\cdot\|F(y(s)+P_N O_s(\omega))\|_W\, ds\\
&\quad+\int_0^{t_1}\|P_N\left(S_{(t_2-s)} - S_{(t_1-s)}\right)\|_{L(W,V)}\cdot\|F(y(s)+P_N O_s(\omega))\|_W\, ds\\
&\leq \int_{t_1}^{t_2} C(t_2-s)^{-\alpha}\, ds\cdot\sup_{0\leq s\leq T}\|F(y(s)+P_N O_s(\omega))\|_W\\
&\quad+\int_0^{t_1}\|P_N\left(S_{(t_2-s)} - S_{(t_1-s)}\right)\|_{L(W,V)}\, ds\cdot\sup_{0\leq s\leq T}\|F(y(s)+P_N O_s(\omega))\|_W
\end{aligned}$$

and therefore

$$\begin{aligned}
&\|(\Phi^N_\omega y)(t_2) - (\Phi^N_\omega y)(t_1)\|_V\\
&\leq C\Big(\int_0^{t_2-t_1} s^{-\alpha}\, ds + \int_0^{t_1}\|P_N(S_{(t_2-t_1+s)} - S_s)\|_{L(W,V)}\, ds\Big)\cdot\sup_{0\leq s\leq T}\|F(y(s)+P_N O_s(\omega))\|_W\\
&= C\Big(\tfrac{(t_2-t_1)^{(1-\alpha)}}{(1-\alpha)} + \int_0^{t_1}\|P_N(S_{(t_2-t_1+s)} - S_s)\|_{L(W,V)}\, ds\Big)\cdot\sup_{0\leq s\leq T}\|F(y(s)+P_N O_s(\omega))\|_W
\end{aligned}$$

for every $t\in[0,T]$, $y\in C\left([0,T],V\right)$, $\omega\in\Omega$ and every $N\in\mathbb{N}$. This shows that $\Phi^N_\omega\left(y\right)$ is indeed in $C\left([0,T],V\right)$ due to Lebesgue's theorem, and hence $\Phi^N_\omega : V\to V$ is well defined for every $N \in \mathbb{N}$ and every $\omega \in \Omega$. Furthermore, we have

$$
\begin{aligned}
&\|(\Phi_\omega^N y)(t) - (\Phi_\omega^N z)(t)\|_V \\
&\leq \int_0^t \|P_N\, S_{(t-s)}\|_{L(W,V)} \cdot \|F(y(s) + P_N O_s(\omega)) - F(z(s) + P_N O_s(\omega))\|_W \, ds \\
&\leq \int_0^t C \cdot (t-s)^{-\alpha} \cdot C \cdot \|y(s) - z(s)\|_V \, ds \\
&\leq C^2 \int_0^t (t-s)^{-\alpha} e^{-\mu s} \, ds \cdot \sup_{0 \leq s \leq T} (e^{\mu s} \|y(s) - z(s)\|_V)
\end{aligned}
$$

and

$$
e^{\mu t} \|(\Phi_\omega^N y)(t) - (\Phi_\omega^N z)(t)\|_V \leq C^2 \int_0^t \frac{e^{\mu s}}{s^\alpha} \, ds \cdot \|y - z\|_\mu,
$$

which implies

$$
\|\Phi_\omega^N(y) - \Phi_\omega^N(z)\|_\mu \leq C^2 \int_0^T \frac{e^{\mu s}}{s^\alpha} \, ds \cdot \|y - z\|_\mu
$$

for every $t \in [0,T]$, $y, z \in C([0,T], V)$, $\omega \in \Omega$, $N \in \mathbb{N}$ and every $\mu \in \mathbb{R}$.

By letting $\mu \to -\infty$, we see that $\Phi_\omega^N : C([0,T], V) \to C([0,T], V)$ is a contraction for every $N \in \mathbb{N}$ and every $\omega \in \Omega$. Since $(C([0,T], V), \|\cdot\|_\mu)$ is complete for every $\mu \in \mathbb{R}$, we obtain the existence of unique stochastic processes $Y^N : [0,T] \times \Omega \to V$, $N \in \mathbb{N}$, with continuous sample paths, which satisfies $\left(\Phi_\omega^N\, Y^N(\omega)\right)(t) = Y_t^N(\omega)$ for every $t \in [0,T]$, $\omega \in \Omega$ and every $N \in \mathbb{N}$. This means that

$$
Y_t^N(\omega) = \int_0^t P_N\, S_{(t-s)}\, F(Y_s(\omega) + P_N O_s(\omega)) \, ds
$$

for every $t \in [0,T]$, $\omega \in \Omega$ and every $N \in \mathbb{N}$. Therefore, $X^N : [0,T] \times \Omega \to V$, $N \in \mathbb{N}$, defined by $X_t^N(\omega) := Y_t^N(\omega) + P_N\,(O_t(\omega))$ for every $t \in [0,T]$, $\omega \in \Omega$ and every $N \in \mathbb{N}$ are unique stochastic processes with continuous sample paths satisfying

$$
X_t^N(\omega) = \int_0^t P_N\, S_{(t-s)}\, F(X_s^N(\omega)) \, ds + P_N O_t(\omega)
$$

for every $t \in [0,T]$, $\omega \in \Omega$ and every $N \in \mathbb{N}$. Additionally, we obtain

$$
\begin{aligned}
\|X_t^N\|_V &\leq \int_0^t \|P_N\, S_{(t-s)}\, F(X_s^N)\|_V \, ds + \|P_N O_t\|_V \\
&\leq \int_0^t \|P_N\, S_{(t-s)}\|_{L(W,V)} \cdot \|F(X_s^N)\|_W \, ds + C \\
&\leq \int_0^t C \cdot (t-s)^{-\alpha} \cdot \|F(X_s^N)\|_W \, ds + C \\
&\leq C^2 \int_0^t (t-s)^{-\alpha} (1 + \|X_s^N\|_V) \, ds + C
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\|X_t^N\|_V &\leq C^2\Big(\int_0^T s^{-\alpha}\,ds + \int_0^t (t-s)^{-\alpha}\|X_s^N\|_V\,ds\Big) + C \\
&\leq C^2\Big(\tfrac{T^{(1-\alpha)}}{(1-\alpha)} + \int_0^t (t-s)^{-\alpha}\left\|X_s^N\right\|_V\,ds + 1\Big) \\
&\leq C^2(T+2)\tfrac{1}{(1-\alpha)} + C^2\int_0^t (t-s)^{-\alpha}\|X_s^N\|_V\,ds
\end{aligned}
$$

for every $t \in [0,T]$ and every $N \in \mathbb{N}$. Hence, Lemma 5.10 implies

$$
\left\|X_t^N\right\|_V \leq \mathrm{E}_{(1-\alpha)}\left(T\left(C^2\,\Gamma(1-\alpha)\right)^{\frac{1}{(1-\alpha)}}\right)\frac{C^2\,(T+2)}{(1-\alpha)}
$$

for every $t \in [0,T]$ and every $N \in \mathbb{N}$, which finally shows

$$
\sup_{N\in\mathbb{N}}\sup_{0\leq t\leq T}\|X_t^N(\omega)\|_V < \infty
$$

for every $\omega \in \Omega$. □

5.2. Proof of Lemma 4.1. *Proof.* [Proof of Lemma 4.1] Clearly, $S : (0,T] \to L\left(C\left([0,1]^d,\mathbb{R}\right)\right)$ given by Lemma 4.1 is well defined. Moreover, $S : (0,T] \to L\left(C\left([0,1]^d,\mathbb{R}\right)\right)$ is a locally Lipschitz continuous mapping with $\|S_t\|_{L(C([0,1]^d,\mathbb{R}))} \leq 1$ (see Lemma 6 in [16]). Hence, it is sufficient to show

$$
\sup_{N\in\mathbb{N}}\sup_{0<t\leq T}\Big(t^{\left(\frac{\gamma}{2}+\frac{d}{4}\right)}N^{\gamma}\left\|S_t - P_N S_t\right\|_{L(W,V)}\Big) < \infty
$$

for every $\gamma \in (0,\infty)$. To this end we use the notation $\|x\|_2 = \left(x_1^2 + \ldots + x_d^2\right)^{\frac{1}{2}}$ for every $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$. Then, we obtain

$$
\begin{aligned}
\|S_t v - P_N S_t v\|_{C([0,1]^d,\mathbb{R})} &= \|(I-P_N)\,S_t v\|_{C([0,1]^d,\mathbb{R})} \\
&\leq \sum_{i\in\mathbb{N}^d\setminus\{1,\ldots,N\}^d}\left\|e^{-\lambda_i t}\,\langle e_i, v\rangle_{L^2}\,e_i\right\|_{C([0,1]^d,\mathbb{R})} \\
&\leq \sum_{i\in\mathbb{N}^d\setminus\{1,\ldots,N\}^d} e^{-\lambda_i t}\,|\langle e_i, v\rangle_{L^2}|\,2^{\frac{d}{2}} \\
&\leq 2^{\frac{d}{2}}\Big(\sum_{i\in\mathbb{N}^d\setminus\{1,\ldots,N\}^d} e^{-2\lambda_i t}\Big)^{\frac{1}{2}}\Big(\sum_{i\in\mathbb{N}^d}|\langle e_i, v\rangle_{L^2}|^2\Big)^{\frac{1}{2}}
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\|S_t v - P_N S_t v\|_{C([0,1]^d,\mathbb{R})} &\leq 2^{\frac{d}{2}}\Big(\int_{\{x\in\mathbb{R}^d:\|x\|_2\geq N\}} e^{-2\pi^2\|x\|_2^2 t}\,dx\Big)^{\frac{1}{2}}\|v\|_{L^2} \\
&\leq 2^{\frac{d}{2}}\Big(3^d\int_N^\infty e^{-2\pi^2 r^2 t}r^{(d-1)}dr\Big)^{\frac{1}{2}}\|v\|_{L^2} \\
&\leq 6^{\frac{d}{2}}N^{-\gamma}\Big(\int_N^\infty e^{-2\pi^2 r^2 t}r^{(d+2\gamma-1)}dr\Big)^{\frac{1}{2}}\|v\|_{L^2} \\
&= 6^{\frac{d}{2}}N^{-\gamma}\Big(\int_{2\pi N\sqrt{t}}^\infty e^{-\frac{r^2}{2}}|r|^{d+2\gamma-1}dr\Big)^{\frac{1}{2}}\left(\tfrac{1}{2\pi\sqrt{t}}\right)^{\frac{d+2\gamma}{2}}\|v\|_{L^2}
\end{aligned}
$$

for every $v \in L^2\left((0,1)^d, \mathbb{R}\right)$, $t \in (0,T]$, $N \in \mathbb{N}$ and every $\gamma \in [0,\infty)$ due to Lemma 8 in [16]. Hence, we obtain

$$\|S_t v - P_N S_t v\|_{C([0,1]^d,\mathbb{R})} \leq N^{-\gamma}\Big(\int_{-\infty}^{\infty} e^{-\frac{r^2}{2}}\,|r|^{d+2\gamma-1}\,dr\Big)^{\frac{1}{2}} t^{-\frac{d+2\gamma}{4}}\|v\|_{L^2}$$

and thus, since $\|v\|_{L^2} \leq \|v\|_{C([0,1]^d,\mathbb{R})}$,

$$\|S_t - P_N S_t\|_{L(C([0,1]^d,\mathbb{R}))} \leq \Big(\int_{-\infty}^{\infty} e^{-\frac{r^2}{2}}|r|^{d+2\gamma-1}dr\Big)^{\frac{1}{2}} N^{-\gamma}\, t^{-\left(\frac{d}{4}+\frac{\gamma}{2}\right)}$$

for every $t \in (0,T]$, $N \in \mathbb{N}$ and every $\gamma \in [0,\infty)$. ∎

5.3. Proof of Lemma 4.3. Throughout this subsection we use the notation $\|x\|_2 = \left(x_1^2 + \ldots + x_d^2\right)^{\frac{1}{2}}$ for every $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$. We first present some simple estimates, which we need in the proof of Lemma 4.3. The first one is elementary and proved as Lemma 9 in [16].

LEMMA 5.1. *Let $d \in \mathbb{N}$ be arbitrary. Then, we have*

$$\int_{(0,1)^d}\int_{(0,1)^d} \left(\|x-y\|_2\right)^{-\alpha}\,dx\,dy \leq \frac{(3d)^d}{(d-\alpha)}$$

for every $\alpha \in (0,d)$.

The next one is well known and for example proved as Lemma 10 in [16].

LEMMA 5.2. *Let $Y : \Omega \to \mathbb{R}$ be a $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mapping, that is centered and normal distributed. Then, $\mathbb{E}\,|Y|^p \leq p!\,\sigma^p$ for every $p \in \mathbb{N}$ where $\sigma := \sqrt{\mathbb{E}\,|Y|^2}$.*

LEMMA 5.3. *Let $d \in \mathbb{N}$ and let $e_i \in C\left([0,1]^d, \mathbb{R}\right)$, $i \in \mathbb{N}^d$, be given by (4.1). Then, we obtain*

$$|e_i(x) - e_i(y)| \leq 2^{\frac{d}{2}}\pi\|i\|_2\,\|x-y\|_2$$

for every $x, y \in [0,1]^d$ and every $i = (i_1,\ldots,i_d) \in \mathbb{N}^d$.

Proof. [Proof of Lemma 5.3] Firstly, by the celebrated mean value theorem we have

$$|e_i(x) - e_i(y)| \leq \sup_{z\in[0,1]^d}\|\nabla e_i(z)\|_2\|x-y\|_2$$

for every $x, y \in [0,1]^d$. Since

$$\begin{aligned}
&\left|\frac{\partial}{\partial z_k}e_i(z)\right| = \left|\frac{\partial}{\partial z_k}\left(2^{\frac{d}{2}}\sin(i_1\pi z_1)\cdots\sin(i_d\pi z_d)\right)\right| \\
&= 2^{\frac{d}{2}}|\sin(i_1\pi z_1)\cdots\sin(i_{k-1}\pi z_{k-1})|\cdot\left|\tfrac{\partial}{\partial z_k}\sin(i_k\pi z_k)\right|\cdot|\sin(i_{k+1}\pi z_{k+1})\cdots\sin(i_d\pi z_d)| \\
&\leq 2^{\frac{d}{2}}\left|\tfrac{\partial}{\partial z_k}\sin(i_k\pi z_k)\right| = 2^{\frac{d}{2}}i_k\pi\,|\cos(i_k\pi z_k)| \leq 2^{\frac{d}{2}}i_k\pi
\end{aligned}$$

for every $k = 1,\ldots,d$, every $z \in [0,1]^d$ and every $i = (i_1,\ldots,i_d) \in \mathbb{N}^d$, we obtain

$$|e_i(x) - e_i(y)| \leq \sup_{z\in[0,1]^d}\left(|2^{\frac{d}{2}}i_1\pi|^2 + \ldots + |2^{\frac{d}{2}}i_d\pi|^2\right)^{\frac{1}{2}}\|x-y\|_2 = 2^{\frac{d}{2}}\pi\|i\|_2\,\|x-y\|_2$$

for every $x, y \in [0,1]^d$ and every $i = (i_1, \ldots, i_d) \in \mathbb{N}^d$. ◻

LEMMA 5.4. *Let $\beta : [0,T] \times \Omega \to \mathbb{R}$ be a standard Brownian motion. Then, we obtain*

$$\mathbb{E}\Big| \int_0^{t_2} e^{-\lambda(t_2-s)} d\beta_s - \int_0^{t_1} e^{-\lambda(t_1-s)} d\beta_s \Big|^2 \leq \lambda^{(r-1)} (t_2 - t_1)^r$$

for every $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$, every $r \in [0,1]$ and every $\lambda \in (0,\infty)$.

Proof. [Proof of Lemma 5.4] First of all, we have

$$\begin{aligned}
&\mathbb{E}\Big| \int_0^{t_2} e^{-\lambda(t_2-s)} d\beta_s - \int_0^{t_1} e^{-\lambda(t_1-s)} d\beta_s \Big|^2 \\
&= \mathbb{E}\Big| \int_{t_1}^{t_2} e^{-\lambda(t_2-s)} d\beta_s + \Big(e^{-\lambda(t_2-t_1)} - 1\Big) \int_0^{t_1} e^{-\lambda(t_1-s)} d\beta_s \Big|^2 \\
&= \mathbb{E}\Big| \int_{t_1}^{t_2} e^{-\lambda(t_2-s)} d\beta_s \Big|^2 + \mathbb{E}\Big| \Big(e^{-\lambda(t_2-t_1)} - 1\Big) \int_0^{t_1} e^{-\lambda(t_1-s)} d\beta_s \Big|^2 \\
&= \int_{t_1}^{t_2} e^{-2\lambda(t_2-s)}\, ds + \Big(e^{-\lambda(t_2-t_1)} - 1\Big)^2 \cdot \int_0^{t_1} e^{-2\lambda(t_1-s)}\, ds
\end{aligned}$$

and therefore

$$\begin{aligned}
&\mathbb{E}\Big| \int_0^{t_2} e^{-\lambda(t_2-s)} d\beta_s - \int_0^{t_1} e^{-\lambda(t_1-s)} d\beta_s \Big|^2 \\
&= \int_0^{(t_2-t_1)} e^{-2\lambda s}\, ds + (1 - e^{-\lambda(t_2-t_1)})^2 \tfrac{1}{2\lambda}(1 - e^{-2\lambda t}) \\
&\leq \tfrac{1}{2\lambda}(1 - e^{-2\lambda(t_2-t_1)}) + \tfrac{1}{2\lambda}(1 - e^{-\lambda(t_2-t_1)})^2 = \tfrac{1}{\lambda}(1 - e^{-\lambda(t_2-t_1)})
\end{aligned}$$

for every $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$ and every $\lambda > 0$ due to Itô's isometry. This shows

$$\begin{aligned}
&\mathbb{E}\Big| \int_0^{t_2} e^{-\lambda(t_2-s)} d\beta_s - \int_0^{t_1} e^{-\lambda(t_1-s)} d\beta_s \Big|^2 \\
&\leq \frac{(1 - e^{-\lambda(t_2-t_1)})^r}{\lambda} \leq \Big(\sup_{x>0} \tfrac{1}{x}(1 - e^{-x}) \Big)^r \lambda^{r-1} (t_2 - t_1)^r = \lambda^{r-1}(t_2 - t_1)^r
\end{aligned}$$

for every $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, every $r \in [0,1]$ and every $\lambda > 0$, which is the assertion. ◻

After these four very simple lemmata, we present now two lemmata, which are the essential constituents of the proof of Lemma 4.2. The first one ensures the temporal regularity of the constructed process in Lemma 4.2.

LEMMA 5.5. *Let $d \in \mathbb{N}$, let $\beta^i : [0,T] \times \Omega \to \mathbb{R}$, $i \in \mathbb{N}^d$, be a family of independent standard Brownian motions and let $b : \mathbb{N}^d \to \mathbb{R}$ be a given function. Then, we obtain*

$$\left(\mathbb{E}\Big[\sup_{x \in [0,1]^d} \big| O^N_{t_2}(x) - O^N_{t_1}(x) \big|^p \Big] \right)^{\frac{1}{p}} \leq C \Big(\sum_{i \in \{1,\ldots,N\}^d} |b(i)|^2 \|i\|_2^{4\theta + 4\alpha - 2} \Big)^{\frac{1}{2}} |t_2 - t_1|^\theta$$

for every $t_1, t_2 \in [0,T]$, $N \in \mathbb{N}$, $p \in [1,\infty)$ and every α, $\theta \in (0, \frac{1}{2}]$, where $C = C(d,p,\alpha,\theta) > 0$ is a constant only depending on d, p, α and θ and where the stochastic process $O^N : [0,T] \times \Omega \to C\left([0,1]^d, \mathbb{R}\right)$ is given by

$$O^N_t = \sum_{i \in \{1,\ldots,N\}^d} b(i) \cdot \int_0^t e^{-\lambda_i(t-s)} d\beta^i_s \cdot e_i \qquad \mathbb{P} - a.s.$$

for every $t \in [0,T]$ and every $N \in \mathbb{N}$. Here, $e_i \in C\left([0,1]^d, \mathbb{R}\right)$, $i \in \mathbb{N}^d$, and $\lambda_i \in \mathbb{R}$, $i \in \mathbb{N}^d$, are given in (4.1) *and* (4.2)*.*

Proof. [Proof of Lemma 5.5] Throughout the proof, let $\alpha, \theta \in (0, \frac{1}{2}]$, $p, N \in \mathbb{N}$ with $p > \frac{1}{\alpha}$ and $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$ be fixed. In addition, let $C = C(d,p,\alpha,\theta) > 0$ be a constant, which changes from line to line but only depends on d, p, α and θ. We show now Lemma 5.5 for these parameters and the case with a general $p \in [1,\infty)$ follows then from Jensen's inequality. By definition of O^N, we have

$$\begin{aligned}
&\left(O^N_{t_2}(x) - O^N_{t_1}(x)\right) - \left(O^N_{t_2}(y) - O^N_{t_1}(y)\right) \\
&\quad = \sum_{i \in \{1,\ldots,N\}^d} b(i) \left(\int_0^{t_2} e^{-\lambda_i (t_2 - s)} d\beta^i_s - \int_0^{t_1} e^{-\lambda_i (t_1 - s)} d\beta^i_s \right) \left(e_i(x) - e_i(y)\right)
\end{aligned}$$

$\mathbb{P}-$a.s. for every $x, y \in [0,1]^d$. Hence, Lemma 5.3 and Lemma 5.4 yield

$$\begin{aligned}
&\mathbb{E}\left|\left(O^N_{t_2}(x) - O^N_{t_1}(x)\right) - \left(O^N_{t_2}(y) - O^N_{t_1}(y)\right)\right|^2 \\
&= \sum_{i \in \{1,\ldots,N\}^d} |b(i)|^2 \mathbb{E}\left| \int_0^{t_2} e^{-\lambda_i (t_2 - s)} d\beta^i_s - \int_0^{t_1} e^{-\lambda_i (t_1 - s)} d\beta^i_s \right|^2 |e_i(x) - e_i(y)|^2 \\
&\leq \sum_{i \in \{1,\ldots,N\}^d} |b(i)|^2 \lambda_i^{2\theta - 1} (t_2 - t_1)^{2\theta} \cdot \left(2^d \pi^2 \|i\|_2^2 \, \|x - y\|_2^2\right)^{2\alpha} \left(|e_i(x)| + |e_i(y)|\right)^{2(1-2\alpha)} \\
&\leq C (t_2 - t_1)^{2\theta} \|x - y\|_2^{4\alpha} \cdot \sum_{i \in \{1,\ldots,N\}^d} |b(i)|^2 \left(\pi^2 \|i\|_2^2\right)^{2\theta - 1} \|i\|_2^{4\alpha}
\end{aligned}$$

and therefore

$$\begin{aligned}
&\mathbb{E}\left|\left(O^N_{t_2}(x) - O^N_{t_1}(x)\right) - \left(O^N_{t_2}(y) - O^N_{t_1}(y)\right)\right|^2 \\
&\qquad \leq C (t_2 - t_1)^{2\theta} \|x - y\|_2^{4\alpha} \cdot \sum_{i \in \{1,\ldots,N\}^d} |b(i)|^2 \|i\|_2^{4\theta + 4\alpha - 2} \qquad (5.1)
\end{aligned}$$

for every $x, y \in [0,1]^d$. In addition, we also have

$$\begin{aligned}
\mathbb{E}|O^N_{t_2}(x) - O^N_{t_1}(x)|^2 &= \sum_{i \in \{1,\ldots,N\}^d} |b(i)|^2 \mathbb{E}\left| \int_0^{t_2} e^{-\lambda_i (t_2 - s)} d\beta^i_s - \int_0^{t_1} e^{-\lambda_i (t_1 - s)} d\beta^i_s \right|^2 |e_i(x)|^2 \\
&\leq C \sum_{i \in \{1,\ldots,N\}^d} |b(i)|^2 \lambda_i^{2\theta - 1} (t_2 - t_1)^{2\theta} \\
&\leq C \sum_{i \in \{1,\ldots,N\}^d} |b(i)|^2 \|i\|_2^{4\theta - 2} (t_2 - t_1)^{2\theta} \qquad (5.2)
\end{aligned}$$

for every $x, y \in [0,1]^d$ due to Lemma 5.4. Hence, the Sobolev embedding given in

Theorem 1 in Section 2.2.4 in [29] (see also Section 2.4.4 there) and Lemma 5.2 yield

$$
\begin{aligned}
&\mathbb{E}\left[\|O^N_{t_2} - O^N_{t_1}\|^p_{C([0,1]^d,\mathbb{R})}\right]\\
&\leq C\int_{(0,1)^d}\int_{(0,1)^d} \frac{\mathbb{E}\left|\left(O^N_{t_2}(x) - O^N_{t_1}(x)\right) - \left(O^N_{t_2}(y) - O^N_{t_1}(y)\right)\right|^p}{\|x-y\|_2^{(d+p\alpha)}}dx\,dy\\
&\quad + C\int_{(0,1)^d} \mathbb{E}\left|O^N_{t_2}(x) - O^N_{t_1}(x)\right|^p dx\\
&\leq C\int_{(0,1)^d}\int_{(0,1)^d} \frac{\left(\mathbb{E}\left|\left(O^N_{t_2}(x) - O^N_{t_1}(x)\right) - \left(O^N_{t_2}(y) - O^N_{t_1}(y)\right)\right|^2\right)^{\frac{p}{2}}}{\|x-y\|_2^{(d+p\alpha)}}dx\,dy\\
&\quad + C\int_{(0,1)^d} \left(\mathbb{E}\left|O^N_{t_2}(x) - O^N_{t_1}(x)\right|^2\right)^{\frac{p}{2}} dx
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\mathbb{E}\left[\|O^N_{t_2} - O^N_{t_1}\|^p_{C([0,1]^d,\mathbb{R})}\right]\\
&\leq C\int_{(0,1)^d}\int_{(0,1)^d} \frac{(t_2-t_1)^{2\theta}\|x-y\|_2^{4\alpha})^{\frac{p}{2}}}{\|x-y\|_2^{d+p\alpha}}dx\,dy\cdot\Big(\sum_{i\in\{1,\dots,N\}^d}|b(i)|^2\|i\|_2^{4\theta+4\alpha-2}\Big)^{\frac{p}{2}}\\
&\qquad + C\int_{(0,1)^d}\Big(\sum_{i\in\{1,\dots,N\}^d}|b(i)|^2\|i\|_2^{4\theta-2}(t_2-t_1)^{2\theta}\Big)^{\frac{p}{2}}dx\\
&\leq C\Big(1+\int_{(0,1)^d}\int_{(0,1)^d}\|x-y\|_2^{p\alpha-d}dx\,dy\Big)(t_2-t_1)^{p\theta}\Big(\sum_{i\in\{1,\dots,N\}^d}|b(i)|^2\|i\|_2^{(4\theta+4\alpha-2)}\Big)^{\frac{p}{2}}
\end{aligned}
$$

due to (5.1) and (5.2). Due to Lemma 5.1 this shows

$$
\Big(\mathbb{E}\,\|O^N_{t_2} - O^N_{t_1}\|^p_{C([0,1]^d,\mathbb{R})}\Big)^{\frac{1}{p}} \leq C\Big(\sum_{i\in\{1,\dots,N\}^d}|b(i)|^2\|i\|_2^{(4\theta+4\alpha-2)}\Big)^{\frac{1}{2}}(t_2-t_1)^{\theta}\ ,
$$

which is the assertion. ☐

The following lemma ensures the spatial regularity of the constructed process from Lemma 4.2.

LEMMA 5.6. *Let $d\in\mathbb{N}$, let $\beta^i:[0,T]\times\Omega\to\mathbb{R}$, $i\in\mathbb{N}^d$, be a family of independent standard Brownian motions and let $b:\mathbb{N}^d\to\mathbb{R}$ be a given function. Then, we obtain*

$$
\left(\mathbb{E}\left[\sup_{0\leq t\leq T}\sup_{x\in[0,1]^d}\left|O^N_t(x) - O^M_t(x)\right|^p\right]\right)^{\frac{1}{p}} \leq C\left(\sum_{i\in\{1,\dots,N\}^d\setminus\{1,\dots,M\}^d}|b(i)|^2\|i\|_2^{(8\alpha-2)}\right)^{\frac{1}{2}}
$$

for every $N,M\in\mathbb{N}$ with $N\geq M$, every $p\in[1,\infty)$, $\alpha\in\left(0,\frac{1}{2}\right)$, where $C=C(d,p,\alpha,T)>0$ is a constant depending only on d, p, α and T and where the stochastic process $O^N:[0,T]\times\Omega\to C\left([0,1]^d,\mathbb{R}\right)$ is given by

$$
O^N_t(\omega) = \sum_{i\in\{1,\dots,N\}^d} b(i)\Big(-\lambda_i\int_0^t e^{-\lambda_i(t-s)}\beta^i_s(\omega)\,ds + \beta^i_t(\omega)\Big)e_i
$$

for every $t \in [0,T]$, $\omega \in \Omega$ *and every* $N \in \mathbb{N}$. *Here,* $e_i \in C\left([0,1]^d, \mathbb{R}\right)$, $i \in \mathbb{N}^d$, *and* $\lambda_i \in \mathbb{R}$, $i \in \mathbb{N}^d$, *are given in* (4.1) *and* (4.2).

Proof. [Proof of Lemma 5.6] Throughout the proof, let $\alpha \in (0, \frac{1}{2})$ and $p, N, M \in \mathbb{N}$ with $p > \frac{1}{\alpha}$ and with $N \geq M$ be fixed. In addition, let $C = C(d, p, \alpha, T) > 0$ be a constant, which changes from line to line but only depends on d, p, α and T. As in the proof of Lemma 5.5, we show now the assertion for these parameters and the case with a general $p \in [1, \infty)$ follows then from Jensen's inequality. Then, let $Y^{N,M} : [0,T] \times \Omega \to C\left([0,1]^d, \mathbb{R}\right)$ be a stochastic processes with continuous sample paths given by

$$Y_t^{N,M} = \sum_{\substack{i \in \{1,\dots,N\}^d \\ \setminus \{1,\dots,M\}^d}} b(i) \int_0^t (t-s)^{-\alpha} \, e^{-\lambda_i (t-s)} d\beta_s^i \cdot e_i \qquad \mathbb{P}-\text{a.s.}$$

for every $t \in [0,T]$. One immediately checks, that the processes

$$\int_0^t (t-s)^{-\alpha} e^{\lambda_i (t-s)} d\beta_s^i, \qquad t \in [0,T], \qquad i \in \mathbb{N}^d,$$

are mean square Hölder continuous and indeed have a modification with continuous sample paths due to Kolmogorov's theorem (see e.g. Theorem 3.3 in [6]). Then, the factorization method (see e.g. Section 5.3 in [6] or also Section 5 in [3]) yields

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T} \left\| O_t^N - O_t^M \right\|_{C([0,1]^d, \mathbb{R})}^p &= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \frac{\sin(\pi\alpha)}{\pi} \int_0^t (t-s)^{\alpha-1} S_{(t-s)} Y_s^{N,M} ds \right\|_{C([0,1]^d, \mathbb{R})}^p \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t (t-s)^{\alpha-1} S_{(t-s)} Y_s^{N,M} ds \right\|_{C([0,1]^d, \mathbb{R})}^p \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T} \left(\int_0^t \left\| (t-s)^{\alpha-1} S_{(t-s)} Y_s^{N,M} \right\|_{C([0,1]^d, \mathbb{R})} ds \right)^p
\end{aligned}$$

and therefore

$$\begin{aligned}
&\mathbb{E} \sup_{0 \leq t \leq T} \| O_t^N - O_t^M \|_{C([0,1]^d, \mathbb{R})}^p \\
&\leq \left(\sup_{0 \leq t \leq T} \| S_t \|_{L(C([0,1]^d, \mathbb{R}))} \right)^p \cdot \mathbb{E} \sup_{0 \leq t \leq T} \left(\int_0^t (t-s)^{(\alpha-1)} \| Y_s^{N,M} \|_{C([0,1]^d, \mathbb{R})} ds \right)^p \\
&\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \left\{ \left(\int_0^t (t-s)^{\frac{p(\alpha-1)}{p-1}} ds \right)^{p-1} \cdot \int_0^t \| Y_s^{N,M} \|_{C([0,1]^d, \mathbb{R})}^p ds \right\} \right] \\
&= \mathbb{E} \left[\left(\int_0^T s^{\frac{p(\alpha-1)}{p-1}} ds \right)^{(p-1)} \cdot \int_0^T \| Y_s^{N,M} \|_{C([0,1]^d, \mathbb{R})}^p ds \right] \\
&\leq C \cdot \int_0^T \mathbb{E} \| Y_s^{N,M} \|_{C([0,1]^d, \mathbb{R})}^p ds
\end{aligned}$$

due to Hölder's inequality and since $\sup_{0 \leq t \leq T} \| S_t \|_{L(C([0,1]^d, \mathbb{R}))} \leq 1$. This shows

$$\mathbb{E} \sup_{0 \leq t \leq T} \left\| O_t^N - O_t^M \right\|_{C([0,1]^d, \mathbb{R})}^p \leq C \sup_{0 \leq s \leq T} \mathbb{E} \left\| Y_s^{N,M} \right\|_{C([0,1]^d, \mathbb{R})}^p$$

and hence

$$\left(\mathbb{E} \sup_{0 \leq t \leq T} \| O_t^N - O_t^M \|_{C([0,1]^d, \mathbb{R})}^p \right)^{\frac{1}{p}} \leq C \sup_{0 \leq t \leq T} \left(\mathbb{E} \| Y_t^{N,M} \|_{C([0,1]^d, \mathbb{R})}^p \right)^{\frac{1}{p}}. \tag{5.3}$$

Hence, it remains to estimate the expression on the right hand side of (5.3). For this, we denote $\mathcal{I}_N := \{1, 2, \ldots, N\}^d$ and $\mathcal{I}_M := \{1, 2, \ldots, M\}^d$ and then, we have

$$\begin{aligned}
&\mathbb{E}|Y_t^{N,M}(x) - Y_t^{N,M}(y)|^2 \\
&= \mathbb{E}\Big| \sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M} b(i) \int_0^t (t-s)^{-\alpha} e^{-\lambda_i(t-s)} d\beta_s^i \cdot (e_i(x) - e_i(y))\Big|^2 \\
&= \sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M} |b(i)|^2 \mathbb{E}\Big| \int_0^t (t-s)^{-\alpha} e^{-\lambda_i(t-s)} d\beta_s^i\Big|^2 |e_i(x) - e_i(y)|^2 \\
&= \sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M} |b(i)|^2 \int_0^t s^{-2\alpha} e^{-2\lambda_i s} ds \cdot |e_i(x) - e_i(y)|^{4\alpha} |e_i(x) - e_i(y)|^{2(1-2\alpha)} \\
&\leq C \sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M} |b(i)|^2 \int_0^\infty s^{-2\alpha} e^{-s} ds \cdot (2\lambda_i)^{2\alpha-1} \left(2^d \pi^2 \|i\|_2^2 \|x-y\|_2^2\right)^{2\alpha}
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathbb{E}|Y_t^{N,M}(x) - Y_t^{N,M}(y)|^2 &\leq C \sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M} |b(i)|^2 \left(2\pi^2\|i\|_2^2\right)^{(2\alpha-1)} \|i\|_2^{4\alpha} \|x-y\|_2^{4\alpha} \\
&\leq C \sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M} |b(i)|^2 \|i\|_2^{(8\alpha-2)} \cdot \|x-y\|_2^{4\alpha} \qquad (5.4)
\end{aligned}$$

for every $t \in [0, T]$ and every $x, y \in [0, 1]^d$. In addition, we have

$$\begin{aligned}
\mathbb{E}|Y_t^{N,M}(x)|^2 &= \mathbb{E}\Big| \sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M} b(i) \int_0^t (t-s)^{-\alpha} e^{-\lambda_i(t-s)} d\beta_s^i \cdot e_i(x)\Big|^2 \\
&= \sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M} |b(i)|^2 \cdot \mathbb{E}\Big| \int_0^t (t-s)^{-\alpha} e^{-\lambda_i(t-s)} d\beta_s^i\Big|^2 |e_i(x)|^2 \\
&= \sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M} |b(i)|^2 \int_0^t s^{-2\alpha} e^{-2\lambda_i s} ds \cdot |e_i(x)|^2
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathbb{E}\left|Y_t^{N,M}(x)\right|^2 &= \sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M} |b(i)|^2 \int_0^{2t\lambda_i} s^{-2\alpha} e^{-s} ds \cdot (2\lambda_i)^{2\alpha-1} |e_i(x)|^2 \\
&\leq C \sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M} |b(i)|^2 \int_0^\infty s^{-2\alpha} e^{-s} ds \cdot \|i\|_2^{4\alpha-2} |e_i(x)|^2 \\
&\leq C \sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M} |b(i)|^2 \|i\|_2^{(8\alpha-2)} \qquad (5.5)
\end{aligned}$$

for every $t \in [0, T]$ and every $x, y \in [0, 1]^d$. Hence, again due to the Sobolev embedding given in Theorem 1 in Section 2.2.4 in [29] (see also Section 2.4.4 there) and Lemma

5.2, we obtain

$$
\begin{aligned}
\sup_{0\le t\le T}\mathbb{E}\|Y_t^{N,M}\|^p_{C([0,1]^d,\mathbb{R})} &\le C \sup_{0\le t\le T}\Big(\int_{(0,1)^d}\int_{(0,1)^d}\frac{\mathbb{E}|Y_t^{N,M}(x)-Y_t^{N,M}(y)|^p}{\|x-y\|_2^{d+p\alpha}}dx\,dy\Big)\\
&\quad + C\sup_{0\le t\le T}\Big(\int_{(0,1)^d}\mathbb{E}|Y_t^{N,M}(x)|^p dx\Big)\\
&\le C\sup_{0\le t\le T}\int_{(0,1)^d}\int_{(0,1)^d}\frac{\Big(\mathbb{E}|Y_t^{N,M}(x)-Y_t^{N,M}(y)|^2\Big)^{\frac{p}{2}}}{\|x-y\|_2^{d+p\alpha}}dx\,dy\\
&\quad + C\sup_{0\le t\le T}\int_{(0,1)^d}\Big(\mathbb{E}|Y_t^{N,M}(x)|^2\Big)^{\frac{p}{2}}dx
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\sup_{0\le t\le T}\Big(\mathbb{E}\|Y_t^{N,M}\|^p_{C([0,1]^d,\mathbb{R})}\Big)^{\frac{1}{p}}\\
&\le \Big(C\int_{(0,1)^d}\int_{(0,1)^d}\frac{\Big(\sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M}|b(i)|^2\|i\|_2^{8\alpha-2}\cdot\|x-y\|_2^{4\alpha}\Big)^{\frac{p}{2}}}{\|x-y\|^{d+p\alpha}}dx\,dy\Big)^{\frac{1}{p}}\\
&\quad + C\Big(\sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M}|b(i)|^2\|i\|_2^{8\alpha-2}\Big)^{\frac{1}{2}}\\
&\le C\Big(1+\int_{(0,1)^d}\int_{(0,1)^d}\|x-y\|^{p\alpha-d}dx\,dy\Big)^{\frac{1}{p}}\cdot\Big(\sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M}|b(i)|^2\|i\|_2^{8\alpha-2}\Big)^{\frac{1}{2}}\\
&\le C\Big(\sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M}|b(i)|^2\|i\|_2^{8\alpha-2}\Big)^{\frac{1}{2}}
\end{aligned}
$$

due to (5.4) and (5.5) and Lemma 5.1. Therefore, (5.3) yields

$$
\Big(\mathbb{E}\sup_{0\le t\le T}\big\|O_t^N-O_t^M\big\|^p_{C([0,1]^d,\mathbb{R})}\Big)^{\frac{1}{p}}\le C\Big(\sum_{i\in\mathcal{I}_N\setminus\mathcal{I}_M}|b(i)|^2\|i\|_2^{8\alpha-2}\Big)^{\frac{1}{2}},
$$

which is the assertion. □

Finally, we present the missing proof of Lemma 4.2.

Proof. [Proof of Lemma 4.2] In this proof we use the stochastic processes $O^N : [0,T]\times\Omega\to C\left([0,1]^d,\mathbb{R}\right)$ given by

$$
O_t^N(\omega) := \sum_{i\in\{1,\dots,N\}^d} b(i)\Big(-\lambda_i\int_0^t e^{-\lambda_i(t-s)}\beta_s^i(\omega)\,ds+\beta_t^i(\omega)\Big)\cdot e_i
$$

for every $\omega\in\Omega$, every $t\in[0,T]$ and every $N\in\mathbb{N}$. Note that by definition of O^N, we have

$$
O_t^N = \sum_{i\in\{1,\dots,N\}^d} b(i)\int_0^t e^{-\lambda_i(t-s)}d\beta_s^i\cdot e_i \qquad \mathbb{P}-\text{a.s.}
$$

for every $t\in[0,T]$ and every $N\in\mathbb{N}$. Note also that the space of continuous functions from $[0,T]$ to $C\left([0,1]^d,\mathbb{R}\right)$, i.e.

$$
C\left([0,T],C([0,1]^d,\mathbb{R})\right)=C\left([0,T]\times[0,1]^d,\mathbb{R}\right),
$$

equipped with the norm

$$\|f\|_\infty = \sup_{t\in[0,T]} \|f(t)\| = \sup_{t\in[0,T]} \sup_{x\in[0,1]^d} |f(t)(x)|$$

for every $f : [0,T] \to C\left([0,1]^d, \mathbb{R}\right) \in C\left([0,T], C\left([0,1]^d, \mathbb{R}\right)\right)$ is a separable $\mathbb{R}$-Banach space. In this proof we also use the $\mathbb{R}$-Banach spaces

$$\mathcal{V}_p := L^p\left((\Omega, \mathcal{F}, \mathbb{P}), \left(C\left([0,T], C\left([0,1]^d, \mathbb{R}\right)\right), \|\cdot\|_\infty\right)\right)$$

of equivalence classes of $\mathcal{F}/\mathcal{B}\left(C\left([0,T], C\left([0,1]^d, \mathbb{R}\right)\right)\right)$-measurable and p-Bochner integrable functions from Ω to $C([0,T], C([0,1]^d, \mathbb{R}))$ for every $p \in [1,\infty)$ (see Section A in the appendix in [27]). In the following we do as usual not distinguish between stochastic processes and their corresponding equivalence class in $\mathcal{V}_p$, $p \in [1,\infty)$. Note that the norm in $\mathcal{V}_p$ is given by $\left(\mathbb{E}\sup_{0\le t\le T} \|Y_t\|^p_{C([0,1]^d,\mathbb{R})}\right)^{\frac{1}{p}}$ for every $Y \in \mathcal{V}_p$ and every $p \in [1,\infty)$.

Hence, note that $O^N \in \mathcal{V}_p$ for every $N \in \mathbb{N}$ and every $p \in [1,\infty)$. Then, due to Lemma 5.6, we have

$$\begin{aligned}
\Big(\mathbb{E} \sup_{0\le t\le T} \|O^N_t - O^M_t\|^p_{C([0,1]^d,\mathbb{R})}\Big)^{\frac{1}{p}} &\le C(d,p,\alpha,T)\Big(\sum_{i\in\mathbb{N}^d\setminus\{1,\dots,M\}^d} |b(i)|^2 \|i\|_2^{8\alpha-2}\Big)^{\frac{1}{2}} \\
&\le C(d,p,\alpha,T)\Big(\sum_{i\in\mathbb{N}^d\setminus\{1,\dots,M\}^d} |b(i)|^2 \|i\|_2^{2\rho-2}\Big)^{\frac{1}{2}} \cdot M^{4\alpha-\rho} \\
&\le C(d,p,\alpha,T)\Big(\sum_{i\in\mathbb{N}^d} |b(i)|^2 \|i\|_2^{2\rho-2}\Big)^{\frac{1}{2}} \cdot M^{4\alpha-\rho}
\end{aligned}$$

for every $N, M \in \mathbb{N}$ with $N \ge M$, every $p \in [1,\infty)$, every $\alpha \in (0, \min(\frac{1}{2}, \frac{\rho}{4}))$ and with appropriate constants $C(d,p,\alpha,T) > 0$ given in Lemma 5.6. In particular, O^N is a Cauchy sequence in $\mathcal{V}_p$ and hence there exists a stochastic process $\tilde{O} : [0,T] \times \Omega \to C\left([0,1]^d, \mathbb{R}\right)$ with $\tilde{O} \in \mathcal{V}_p$ and

$$\Big(\mathbb{E} \sup_{0\le t\le T} \|\tilde{O}_t - O^M_t\|^p_{C([0,1]^d,\mathbb{R})}\Big)^{\frac{1}{p}} \le C(p,d,\alpha,T)\Big(\sum_{i\in\mathbb{N}^d} |b(i)|^2 \|i\|_2^{2\rho-2}\Big)^{\frac{1}{2}} N^{4\alpha-\rho}$$

for every $N \in \mathbb{N}$, every $p \in [1,\infty)$ and every $\alpha \in (0, \min(\frac{1}{2}, \frac{\rho}{4}))$. This shows

$$\sup_{N\in\mathbb{N}} \Big\{N^\gamma \Big(\mathbb{E} \sup_{0\le t\le T} \|\tilde{O}_t - O^N_t\|^p_{C([0,1]^d,\mathbb{R})}\Big)^{\frac{1}{p}}\Big\} < \infty$$

for every $\gamma \in (0,\rho)$ and every $p \in [1,\infty)$. Hence, we obtain

$$\mathbb{P}\left[\sup_{N\in\mathbb{N}} \Big\{N^\gamma \sup_{0\le t\le T} \|\tilde{O}_t - O^N_t\|_{C([0,1]^d,\mathbb{R})}\Big\} < \infty\right] = 1$$

for every $\gamma \in (0,\rho)$ (see for instance Lemma 1 in [18]). This yields

$$\mathbb{P}\left[\sup_{N\in\mathbb{N}} \sup_{0\le t\le T} \Big\{N^\gamma \|\tilde{O}_t - O^N_t\|_{C([0,1]^d,\mathbb{R})}\Big\} < \infty \;\; \forall \;\; \gamma \in (0,\rho)\right] = 1.$$

In particular, we have

$$\mathbb{P}\left[\lim_{N\to\infty}\sup_{0\le t\le T}\|\tilde{O}_t - O_t^N\|_{C([0,1]^d,\mathbb{R})} = 0\right] = 1 \tag{5.6}$$

and

$$\mathbb{P}\left[\sup_{N\in\mathbb{N}}\sup_{0\le t\le T}\left\{N^\gamma\|\tilde{O}_t - P_N\tilde{O}_t\|_{C([0,1]^d,\mathbb{R})}\right\} < \infty \ \ \forall\ \gamma\in(0,\rho)\right] = 1. \tag{5.7}$$

Due to Lemma 5.5, we also have

$$\begin{aligned}\Big(\mathbb{E}\|O_{t_2}^N - O_{t_1}^N\|_{C([0,1]^d,\mathbb{R})}^p\Big)^{\frac{1}{p}} &\le C(d,p,\rho,\theta)\Big(\sum_{i\in\{1,\dots,N\}^d}|b(i)|^2\|i\|_2^{4\theta+4(\frac{\rho}{2}-\theta)-2}\Big)^{\frac{1}{2}}|t_2-t_1|^\theta \\ &\le C(d,p,\rho,\theta)\Big(\sum_{i\in\mathbb{N}^d}|b(i)|^2\|i\|_2^{2\rho-2}\Big)^{\frac{1}{2}}|t_2-t_1|^\theta\end{aligned}$$

for every $t_1,t_2\in[0,T]$, every $N\in\mathbb{N}$, every $\theta\in(0,\frac{\rho}{2})$, $\theta\le\frac{1}{2}$ and where $C(d,p,\rho,\theta)>0$ are appropriate constants (see Lemma 5.5). This shows

$$\Big(\mathbb{E}\|\tilde{O}_{t_2} - \tilde{O}_{t_1}\|_{C([0,1]^d,\mathbb{R})}^p\Big)^{\frac{1}{p}} \le C(d,p,\rho,\theta)\Big(\sum_{i\in\mathbb{N}^d}|b(i)|^2\|i\|_2^{2\rho-2}\Big)^{\frac{1}{2}}|t_2-t_1|^\theta$$

for every $t_1,t_2\in[0,T]$ and every $\theta\in\big(0,\frac{\rho}{2}\big)$, $\theta\le\frac{1}{2}$. Hence, Kolmogorov's theorem (see e.g. Theorem 3.3 in [6]) yields

$$\mathbb{P}\left[\sup_{0\le t_1<t_2\le T}\frac{\|\tilde{O}_{t_2}-\tilde{O}_{t_1}\|_{C([0,1]^d,\mathbb{R})}}{(t_2-t_1)^\theta} < \infty\right] = 1$$

for every $\theta\in\big(0,\min\big(\frac{1}{2},\frac{\rho}{2}\big)\big)$. Therefore, we obtain

$$\mathbb{P}\left[\sup_{0\le t_1<t_2\le T}\frac{\|\tilde{O}_{t_2}-\tilde{O}_{t_1}\|_{C([0,1]^d,\mathbb{R})}}{(t_2-t_1)^\theta} < \infty\ \ \forall\ \theta\in\big(0,\min\{\tfrac{1}{2},\tfrac{\rho}{2}\}\big)\right] = 1. \tag{5.8}$$

Hence, equation (5.8) and equation (5.7) show the existence of a stochastic process $O:[0,T]\times\Omega\to C\left([0,1]^d,\mathbb{R}\right)$, which satisfies

$$\sup_{0\le t_1<t_2\le T}\frac{\|O_{t_2}(\omega)-O_{t_1}(\omega)\|_{C([0,1]^d,\mathbb{R})}}{(t_2-t_1)^\theta} < \infty$$

and

$$\sup_{N\in\mathbb{N}}\sup_{0\le t\le T}\Big(\|O_t(\omega)-P_NO_t(\omega)\|_{C([0,1]^d,\mathbb{R})}\,N^\gamma\Big) < \infty$$

for every $\omega\in\Omega$, every $\theta\in\big(0,\min\big(\frac{1}{2},\frac{\rho}{2}\big)\big)$ and every $\gamma\in(0,\rho)$ and which is indistinguishable from $\tilde{O}$, i.e.

$$\mathbb{P}\Big[\ O_t = \tilde{O}_t\ \ \forall\ t\in[0,T]\ \Big] = 1.$$

This completes the proof of Lemma 4.2. $\square$

5.4. Proof of Lemma 4.6. *Proof.* [Proof of Lemma 4.6] First of all, we have

$$\begin{aligned}\sum_{n=1}^{\infty} n^{2+2\gamma}\, e^{-2n^2\pi^2 t} &\leq \int_0^{\infty} (x+1)^{2+2\gamma}\, e^{-2x^2\pi^2 t}\, dx \\ &\leq \int_0^{\infty} 8\left(x^{2+2\gamma}+1\right) e^{-2x^2\pi^2 t}\, dx \\ &= \frac{1}{2\pi\sqrt{t}} \int_0^{\infty} 8\left(\tfrac{x^{2+2\gamma}}{(2\pi\sqrt{t})^{2+2\gamma}}+1\right) e^{-\frac{x^2}{2}}\, dx\ ,\end{aligned}$$

and hence

$$\begin{aligned}\sum_{n=1}^{\infty} n^{2+2\gamma}\, e^{-2n^2\pi^2 t} &\leq \frac{4}{\pi\sqrt{t}} \int_0^{\infty} \left(\frac{x^{2+2\gamma}}{t^{1+\gamma}}+1\right) e^{-\frac{x^2}{2}}\, dx \\ &\leq \frac{4}{\pi\sqrt{t}} \int_0^{\infty} \left(\frac{x^4+1}{t^{1+\gamma}}+\frac{T^{1+\gamma}}{t^{1+\gamma}}\right) e^{-\frac{x^2}{2}}\, dx \\ &= \frac{4\sqrt{2\pi}}{t^{\left(\frac{3}{2}+\gamma\right)}\pi} \int_0^{\infty} \frac{x^4+1+T^{1+\gamma}}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}\, dx\end{aligned}$$

and finally

$$\sum_{n=1}^{\infty} n^{2+2\gamma}\, e^{-2n^2\pi^2 t} \leq \frac{4\sqrt{2\pi}}{t^{\left(\frac{3}{2}+\gamma\right)}\pi} \int_{\mathbb{R}} \frac{\left(x^4+T^2+2\right)}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \leq \frac{4\left(T^2+5\right)}{t^{\frac{3}{2}+\gamma}}$$

for every $t \in (0,T]$ and every $\gamma \in [0,\frac{1}{2})$. Therefore, we obtain

$$\begin{aligned}&\sup_{0\leq x\leq 1} \Big\{ \sum_{n=N}^{\infty} 2\cdot e^{-n^2\pi^2 t} \cdot |w(\sin(n\pi(\cdot)))| \cdot |\sin(n\pi x)| \Big\} \\ &\leq 2\sum_{n=N}^{\infty} e^{-n^2\pi^2 t} |w(\sin(n\pi(\cdot)))| = \pi\sqrt{2} \sum_{n=N}^{\infty} n\, e^{-n^2\pi^2 t}\, \frac{|w(\sqrt{2}\sin(n\pi(\cdot)))|}{n\pi} \\ &\leq \pi\sqrt{2}\Big(\sum_{n=N}^{\infty} n^2 e^{-2n^2\pi^2 t}\Big)^{\frac{1}{2}} \cdot \Big(\sum_{n=1}^{\infty} \frac{|w(\sqrt{2}\sin(n\pi(\cdot)))|^2}{n^2\pi^2}\Big)^{\frac{1}{2}} \\ &= \pi\sqrt{2}\Big(\sum_{n=N}^{\infty} n^2 e^{-2n^2\pi^2 t}\Big)^{\frac{1}{2}} \|w\|_{H^{-1}} \leq \pi\sqrt{2}\, N^{-\gamma}\Big(\sum_{n=N}^{\infty} n^{2+2\gamma} e^{-2n^2\pi^2 t}\Big)^{\frac{1}{2}} \|w\|_{H^{-1}} \\ &\leq \pi\sqrt{2}\, N^{-\gamma}\left(4\left(T^2+5\right) t^{-\frac{3}{2}-\gamma}\right)^{\frac{1}{2}} \|w\|_{H^{-1}}\end{aligned}$$

and thus

$$\sup_{0\leq x\leq 1} \Big\{ \sum_{n=N}^{\infty} 2\cdot e^{-n^2\pi^2 t} \cdot |w(\sin(n\pi(\cdot)))| \cdot |\sin(n\pi x)| \Big\} \leq \frac{10\,(T+3)}{t^{\left(\frac{3}{4}+\frac{\gamma}{2}\right)} N^{\gamma}} \|w\|_{H^{-1}}$$

for every $w \in H^{-1}((0,1),\mathbb{R})$, $N \in \mathbb{N}$, $\gamma \in [0,\frac{1}{2})$ and every $t \in (0,T]$ due to Lemma 4.7. This implies

$$\|S_t(w)\|_{C([0,1],\mathbb{R})} \leq \sup_{0\leq x\leq 1} \Big\{ \sum_{n=1}^{\infty} 2\cdot e^{-n^2\pi^2 t}\cdot|w(\sin(n\pi(\cdot)))|\cdot|\sin(n\pi x)| \Big\} \leq \frac{10(T+3)}{t^{\frac{3}{4}}} \|w\|_{H^{-1}}$$

and

$$\|S_t(w)-P_N S_t(w)\|_{C([0,1],\mathbb{R})} \leq \sup_{0\leq x\leq 1}\left\{\sum_{n=N+1}^{\infty} 2\cdot e^{-n^2\pi^2 t}\cdot|w(\sin(n\pi(\cdot)))|\cdot|\sin(n\pi x)|\right\}$$
$$\leq \frac{10(T+3)}{t^{\frac{3}{4}+\frac{\gamma}{2}}(N+1)^{\gamma}}\|w\|_{H^{-1}} \leq \frac{10(T+3)}{t^{\frac{3}{4}+\frac{\gamma}{2}}N^{\gamma}}\|w\|_{H^{-1}}$$

for every $t \in (0,T]$, $w \in H^{-1}((0,1),\mathbb{R})$, $\gamma \in [0,\frac{1}{2})$ and every $N \in \mathbb{N}$. Finally, we obtain

$$\sup_{0<t\leq T}\left(t^{\frac{3}{4}}\|S_t\|_{L\left(H^{-1}((0,1),\mathbb{R}),\, C([0,1],\mathbb{R})\right)}\right) < \infty$$

and

$$\sup_{N\in\mathbb{N}}\sup_{0<t\leq T}\left(t^{\frac{3}{4}+\frac{\gamma}{2}}N^{\gamma}\|S_t - P_N S_t\|_{L\left(H^{-1}((0,1),\mathbb{R}),\, C([0,1],\mathbb{R})\right)}\right) < \infty$$

for every $\gamma \in [0,\frac{1}{2})$, which is the assertion. □

5.5. Proof of Lemma 4.7. *Proof.* [Proof of Lemma 4.7] First of all, we have

$$|(\partial v)\,\varphi| = \left|\int_0^1 v(x)\,\varphi'(x)\,dx\right| \leq \int_0^1 |v(x)|\cdot|\varphi'(x)|\;dx \leq \|v\|_{L^2}\cdot\|\varphi\|_{H_0^1}$$

for every $v \in L^2((0,1),\mathbb{R})$ and every $\varphi \in H_0^1((0,1),\mathbb{R})$ and therefore is $\partial : L^2((0,1),\mathbb{R}) \to H_0^1((0,1),\mathbb{R})$ a well defined bounded linear mapping from $L^2((0,1),\mathbb{R})$ to $H_0^1((0,1),\mathbb{R})$ with $\|\partial v\|_{H^{-1}} \leq \|v\|_{L^2}$ for all $v \in L^2((0,1),\mathbb{R})$. Additionally, consider the linear mappings $\Psi : H_0^1((0,1),\mathbb{R}) \to \dot{L}^2((0,1),\mathbb{R})$ and $\Phi : H_0^1((0,1),\mathbb{R}) \to H^{-1}((0,1),\mathbb{R})$ given by $\Psi(v) := v'$ and $(\Phi(v))(\varphi) = \langle v,\varphi\rangle_{H_0^1}$ for every $v,\varphi \in H_0^1((0,1),\mathbb{R})$. By definition Ψ and Φ are linear bijective isometries. Moreover, we have

$$\begin{aligned}\left(\partial[-\Psi\left(\Phi^{-1}(w)\right)]\right)(\varphi) &= -(\partial[\Psi(\Phi^{-1}(w))])(\varphi)\\ &= \langle\Psi(\Phi^{-1}(w)),\varphi'\rangle_{L^2} = \langle(\Phi^{-1}(w))',\varphi'\rangle_{L^2}\\ &= \langle\Phi^{-1}(w),\varphi\rangle_{H_0^1} = (\Phi(\Phi^{-1}(w)))(\varphi) = w(\varphi)\end{aligned}$$

for every $\varphi \in H_0^1((0,1),\mathbb{R})$ and every $w \in H^{-1}((0,1),\mathbb{R})$. This implies

$$\partial(-\Psi(\Phi^{-1}(w))) = w$$

for every $w \in H^{-1}((0,1),\mathbb{R})$ and hence we obtain that $\partial|_{\dot{L}^2((0,1),\mathbb{R})} : \dot{L}^2((0,1),\mathbb{R}) \to H^{-1}((0,1),\mathbb{R})$ is a linear bijective isometry from $\dot{L}^2((0,1),\mathbb{R})$ to $H^{-1}((0,1),\mathbb{R})$. In particular $\partial : L^2((0,1),\mathbb{R}) \to H^{-1}((0,1),\mathbb{R})$ is surjective with $\partial(\dot{L}^2((0,1),\mathbb{R})) = H^{-1}((0,1),\mathbb{R})$ and $\|\partial v\|_{H^{-1}} = \|v\|_{L^2}$ for every $v \in \dot{L}^2((0,1),\mathbb{R})$. Finally, we want to compute $\|w\|_{H^{-1}}$ for $w \in H^{-1}((0,1),\mathbb{R})$. To this end let $w \in H^{-1}((0,1),\mathbb{R})$ be arbitrary and let $v \in \dot{L}^2((0,1),\mathbb{R})$ be the unique element in $\dot{L}^2((0,1),\mathbb{R})$, which satisfies $\partial v = w$. Then, we obtain

$$
\begin{aligned}
\|w\|_{H^{-1}} = \|\partial v\|_{H^{-1}} = \|v\|_{L^2} &= \sum_{n=1}^{\infty} \Big| \int_0^1 \sqrt{2} \cdot \cos(n\pi x) \cdot v(x)\, dx \Big|^2 \\
&= \sum_{n=1}^{\infty} \Big| \int_0^1 \frac{\sqrt{2} \cdot \big(\frac{\partial}{\partial x} \sin(n\pi x)\big)}{n\pi} \cdot v(x)\, dx \Big|^2 \\
&= \sum_{n=1}^{\infty} \frac{| \int_0^1 \sqrt{2} \cdot \big(\frac{\partial}{\partial x} \sin(n\pi x)\big) \cdot v(x) dx|^2}{n^2\pi^2} \\
&= \sum_{n=1}^{\infty} \frac{\big|\langle v, (\sqrt{2}\sin(n\pi(\cdot)))' \rangle_{L^2}\big|^2}{n^2\pi^2} = \sum_{n=1}^{\infty} \frac{|(\partial v)(\sqrt{2}\sin(n\pi(\cdot)))|^2}{n^2\pi^2} \\
&= \sum_{n=1}^{\infty} \frac{|w(\sqrt{2}\sin(n\pi(\cdot)))|^2}{n^2\pi^2},
\end{aligned}
$$

which shows the assertion. □

5.6. Proof of Lemma 4.9. In this subsection we use the finite dimensional $\mathbb{R}$-Banach spaces

$$
P_N\Big(C([0,1],\mathbb{R})\Big) = \Big\{ v \in C([0,1],\mathbb{R}) \,\Big|\, \exists\, v_1, \ldots, v_N \in \mathbb{R} : \\
\forall\, x \in [0,1] : v(x) = \sum_{n=1}^{N} v_n \cdot \sin(n\pi x) \Big\}
$$

equipped with the supremum norm $\|v\|_{C([0,1],\mathbb{R})} = \sup_{0 \le x \le 1} |v(x)|$ for every $v \in C([0,1],\mathbb{R})$ and every $N \in \mathbb{N}$. Due to a similar fixpoint argument as in the proof of Proposition 4.4, Lemma 4.9 follows from the following lemma.

LEMMA 5.7. *Let $\tau \in (0,T], N \in \mathbb{N}$ and let $x^N : [0,\tau] \to P_N(C([0,1],\mathbb{R}))$ and $o^N : [0,\tau] \to P_N(C([0,1],\mathbb{R}))$ be two continuous functions, which satisfy*

$$
x^N(t) = \int_0^t P_N\, S_{(t-s)}\, F(x^N(s)) ds + o^N(t)
$$

for every $t \in [0,\tau]$. Then, we obtain

$$
\sup_{0 \le t \le \tau} \|x^N(t)\|_{C([0,1],\mathbb{R})} \le \exp\Big(16\,\big(c^2+1\big)\,(T+1)\Big(\sup_{0 \le t \le \tau} \|o^N(t)\|^2_{C([0,1],\mathbb{R})} + 1\Big)\Big),
$$

where $c \in \mathbb{R}$ is used in Lemma 4.8.

Proof. [Proof of Lemma 5.7] Firstly, we have

$$
\begin{aligned}
(S_t - I)v &= \sum_{n=1}^{N} 2(e^{-n^2\pi^2 t} - 1) \cdot \int_0^1 \sin(n\pi u)\, v(u)\, du \cdot \sin(n\pi\cdot) \\
&= \sum_{n=1}^{N} 2 \int_0^t \left(-n^2\pi^2\right) e^{-n^2\pi^2 s} ds \cdot \int_0^1 \sin(n\pi u)\, v(u)\, du \cdot \sin(n\pi\cdot) \\
&= \int_0^t \sum_{n=1}^{N} 2e^{-n^2\pi^2 s} \int_0^1 \Big(\frac{\partial^2}{\partial u^2} \sin(n\pi u)\Big) v(u)\, du \cdot \sin(n\pi\cdot) ds \\
&= \int_0^t \sum_{n=1}^{N} 2e^{-n^2\pi^2 s} \int_0^1 \sin(n\pi u)\, v''(u)\, du \cdot \sin(n\pi\cdot) ds \\
&= \int_0^t S_s(v'') ds
\end{aligned}
$$

for every $t \in (0,T]$ and every $v \in P_N(C([0,1],\mathbb{R}))$, which yields

$$
(S_t - I)v = \int_0^t S_s(v'') ds \quad \text{and} \quad \|(S_t - I)v\|_{C([0,1],\mathbb{R})} \leq t \cdot \|v''\|_{C([0,1],\mathbb{R})} . \tag{5.9}
$$

In the next step we define the continuous function $y^N : [0,\tau] \to P_N(C([0,1],\mathbb{R}))$ by

$$
y^N(t) := x^N(t) - o^N(t) = \int_0^t P_N\, S_{(t-s)}\, F(x^N(s)) ds = P_N \int_0^t S_{(t-s)}\, F(x^N(s))\, ds
$$

for every $t \in [0,\tau]$. We also use $w_N : [0,\tau] \to [0,\infty)$ given by

$$
\begin{aligned}
&w_N(h) := \\
&\sup\left\{ \left\| P_N F\left(x^N(t_2)\right) - P_N F\left(x^N(t_1)\right) \right\|_{C([0,1],\mathbb{R})} \in [0,\infty) \,\middle|\, t_1, t_2 \in [0,\tau], |t_1 - t_2| \leq h \right\} \\
&\quad + \sup\left\{ \left\| \frac{\partial^2}{\partial x^2} \left(y^N(t_2) - y^N(t_1)\right) \right\|_{C([0,1],\mathbb{R})} \in [0,\infty) \,\middle|\, t_1, t_2 \in [0,\tau], |t_1 - t_2| \leq h \right\}
\end{aligned}
$$

for every $h \in [0,\tau]$. Note that $w_N : [0,\tau] \to [0,\infty)$ is monoton increasing with $w_N(0) = 0$ and $\lim_{h \searrow 0} w_N(h) = 0$. Moreover, we have

$$
\begin{aligned}
&\frac{y^N(t_2) - y^N(t_1)}{t_2 - t_1} \\
&= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} S_{(t_2-s)} P_N F(x^N(s)) ds + \frac{1}{t_2 - t_1} \int_0^{t_1} (S_{(t_2-s)} - S_{(t_1-s)}) P_N F(x^N(s)) ds \\
&= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} S_{(t_2-s)} P_N F(x^N(s)) ds + \frac{1}{t_2 - t_1} (S_{(t_2-t_1)} - I) y^N(t_1)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\Big\| \frac{y^N(t_2) - y^N(t_1)}{t_2 - t_1} - \frac{\partial^2}{\partial x^2}\, y^N(t_j) - P_N F(x^N(t_j)) \Big\|_{C([0,1],\mathbb{R})} \\
&\leq \Big\| \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} S_{(t_2-s)} P_N F(x^N(s)) ds - P_N F(x^N(t_j)) \Big\|_{C([0,1],\mathbb{R})} \\
&\quad + \Big\| \frac{1}{t_2 - t_1} (S_{(t_2-t_1)} - I) y^N(t_1) - \frac{\partial^2}{\partial x^2}\, y^N(t_j) \Big\|_{C([0,1],\mathbb{R})} \\
&\leq \Big\| \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} S_{(t_2-s)} \left(P_N F(x^N(s)) - P_N F(x^N(t_j)) \right) ds \Big\|_{C([0,1],\mathbb{R})} \\
&\quad + \Big\| \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (S_{(t_2-s)} - I) P_N F(x^N(t_j)) ds \Big\|_{C([0,1],\mathbb{R})} \\
&\quad + \Big\| \frac{1}{t_2 - t_1} (S_{(t_2-t_1)} - I) y^N(t_1) - \frac{\partial^2}{\partial x^2}\, y^N(t_j) \Big\|_{C([0,1],\mathbb{R})}
\end{aligned}
$$

for every $0 \leq t_1 < t_2 \leq \tau$ and every $j \in \{1, 2\}$. Hence, we obtain

$$
\begin{aligned}
&\Big\| \frac{y^N(t_2) - y^N(t_1)}{t_2 - t_1} - \frac{\partial^2}{\partial x^2}\, y^N(t_j) - P_N F(x^N(t_j)) \Big\|_{C([0,1],\mathbb{R})} \\
&\leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \big\| P_N F(x^N(s)) - P_N F(x^N(t_j)) \big\|_{C([0,1],\mathbb{R})} \, ds \\
&\quad + \frac{1}{t_2 - t_1} \int_0^{t_2 - t_1} \big\| (S_s - I)\, P_N F(x^N(t_j)) \big\|_{C([0,1],\mathbb{R})} \, ds \\
&\quad + \Big\| \frac{1}{t_2 - t_1} (S_{(t_2-t_1)} - I) y^N(t_1) - \frac{\partial^2}{\partial x^2}\, y^N(t_1) \Big\|_{C([0,1],\mathbb{R})} + \Big\| \frac{\partial^2}{\partial x^2} \left(y^N(t_1) - y^N(t_j) \right) \Big\|_{C([0,1],\mathbb{R})} \\
&\leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} w_N(|s - t_j|) \, ds + \frac{1}{t_2 - t_1} \int_0^{t_2 - t_1} \int_0^s \Big\| S_u \frac{\partial^2}{\partial x^2}\, P_N F(x^N(t_j)) \Big\|_{C([0,1],\mathbb{R})} du \, ds \\
&\quad + \Big\| \frac{1}{t_2 - t_1} \left(S_{(t_2-t_1)} - I \right) y^N(t_1) - \frac{\partial^2}{\partial x^2}\, y^N(t_1) \Big\|_{C([0,1],\mathbb{R})} + w_N(|t_1 - t_j|)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\| \frac{y^N(t_2) - y^N(t_1)}{(t_2 - t_1)} - \frac{\partial^2}{\partial x^2}\, y^N(t_j) - P_N F(x^N(t_j)) \Big\|_{C([0,1],\mathbb{R})} \\
&\leq 2 \cdot w_N(t_2 - t_1) + (t_2 - t_1) \cdot \Big\| \frac{\partial^2}{\partial x^2}\, P_N F(x^N(t_j)) \Big\|_{C([0,1],\mathbb{R})} \\
&\qquad + \Big\| \frac{1}{t_2 - t_1} \int_0^{t_2 - t_1} S_s \frac{\partial^2}{\partial x^2}\, y^N(t_1) ds - \frac{\partial^2}{\partial x^2}\, y^N(t_1) \Big\|_{C([0,1],\mathbb{R})} \\
&\leq 2 \cdot w_N(t_2 - t_1) + (t_2 - t_1) \cdot \Big\| \frac{\partial^2}{\partial x^2}\, P_N F(x^N(t_j)) \Big\|_{C([0,1],\mathbb{R})} \\
&\qquad + \Big\| \frac{1}{t_2 - t_1} \int_0^{t_2 - t_1} (S_s - I) \frac{\partial^2}{\partial x^2}\, y^N(t_1) ds \Big\|_{C([0,1],\mathbb{R})}
\end{aligned}
$$

for every $0 \leq t_1 < t_2 \leq \tau$ and every $j \in \{1, 2\}$ due to (5.9). This implies

$$\begin{aligned}
&\left\| \frac{y^N(t_2) - y^N(t_1)}{(t_2 - t_1)} - \frac{\partial^2}{\partial x^2} y^N(t_j) - P_N F(x^N(t_j)) \right\|_{C([0,1],\mathbb{R})} \\
&\leq 2 \cdot w_N(t_2 - t_1) + (t_2 - t_1) \cdot \sup_{0 \leq s \leq T} \left\| \frac{\partial^2}{\partial x^2} P_N F(x^N(s)) \right\|_{C([0,1],\mathbb{R})} \\
&\quad + \frac{1}{t_2 - t_1} \int_0^{t_2 - t_1} \left\| (S_s - I) \frac{\partial^2}{\partial x^2} y^N(t_1) \right\|_{C([0,1],\mathbb{R})} ds
\end{aligned}$$

and finally

$$\begin{aligned}
&\left\| \frac{y^N(t_2) - y^N(t_1)}{t_2 - t_1} - \frac{\partial^2}{\partial x^2} y^N(t_j) - P_N F(x^N(t_j)) \right\|_{C([0,1],\mathbb{R})} \\
&\leq 2 \cdot w_N(t_2 - t_1) + (t_2 - t_1) \cdot \sup_{0 \leq s \leq T} \left\| \frac{\partial^2}{\partial x^2} P_N F(x^N(s)) \right\|_{C([0,1],\mathbb{R})} \\
&\quad + (t_2 - t_1) \cdot \left\| \frac{\partial^4}{\partial x^4} y^N(t_1) \right\|_{C([0,1],\mathbb{R})} \\
&\leq 2 \cdot w_N(t_2 - t_1) + (t_2 - t_1) \cdot \sup_{0 \leq s \leq T} \left\| \frac{\partial^2}{\partial x^2} P_N F(x^N(s)) \right\|_{C([0,1],\mathbb{R})} \\
&\quad + (t_2 - t_1) \cdot \sup_{0 \leq s \leq T} \left\| \frac{\partial^4}{\partial x^4} y^N(s) \right\|_{C([0,1],\mathbb{R})}
\end{aligned}$$

for every $0 \leq t_1 < t_2 \leq \tau$ and every $j \in \{1, 2\}$ due to (5.9). This shows

$$\frac{\partial}{\partial t} y^N(t) \;=\; \frac{\partial^2}{\partial x^2} y^N(t) \;+\; P_N F(x^N(t)) \;=\; \frac{\partial^2}{\partial x^2} y^N(t) \;+\; P_N F(y^N(t) \;+\; o^N(t))$$

and hence

$$\begin{aligned}
&\frac{\partial}{\partial t} \|y^N(t)\|_{L^2}^2 \\
&= 2 \langle y^N(t), \frac{\partial}{\partial t} y^N(t) \rangle_{L^2} = 2 \left\langle y^N(t), \frac{\partial^2}{\partial x^2} y^N(t) + P_N F(y^N(t) + o^N(t)) \right\rangle_{L^2} \\
&= 2 \left\langle y^N(t), \frac{\partial^2}{\partial x^2} y^N(t) + F(y^N(t) + o^N(t)) \right\rangle_{L^2} \\
&\leq \|y^N(t)\|_{L^2}^2 \cdot 4c^2 \|o^N(t)\|_{C([0,1],\mathbb{R})}^2 + 4c^2 \|o^N(t)\|_{C([0,1],\mathbb{R})}^4 \\
&\leq \|y^N(t)\|_{L^2}^2 \cdot 4c^2 \sup_{0 \leq s \leq \tau} \|o^N(s)\|_{C([0,1],\mathbb{R})}^2 + 4c^2 \sup_{0 \leq s \leq \tau} \|o^N(s)\|_{C([0,1],\mathbb{R})}^4
\end{aligned}$$

for every $t \in [0, \tau]$ due to Lemma 5.9. Therefore, Gronwall's lemma yields

$$\begin{aligned}
\|y^N(t)\|_{L^2}^2 &\leq \exp\left(4c^2 T \sup_{0 \leq s \leq \tau} \|o^N(s)\|_{C([0,1],\mathbb{R})}^2\right) \cdot \left(2|c| \sup_{0 \leq s \leq \tau} \|o^N(s)\|_{C([0,1],\mathbb{R})}^2\right)^2 \\
&\leq \exp\left(4c^2 T \sup_{0 \leq s \leq \tau} \|o^N(s)\|_{C([0,1],\mathbb{R})}^2\right) \cdot \exp\left(4|c| \sup_{0 \leq s \leq \tau} \|o^N(s)\|_{C([0,1],\mathbb{R})}^2\right) \\
&\leq \exp\left(4(c^2 + 1)(T + 1) \sup_{0 \leq s \leq \tau} \|o^N(s)\|_{C([0,1],\mathbb{R})}^2\right)
\end{aligned}$$

for every $t \in [0, \tau]$ and finally

$$\sup_{0 \le t \le \tau} \|y^N(t)\|_{L^2} \le \exp\Big(4(c^2+1)(T+1) \sup_{0 \le s \le \tau} \|o^N(s)\|^2_{C([0,1],\mathbb{R})}\Big). \tag{5.10}$$

In the next step we obtain

$$\begin{aligned}\|y^N(t)\|_{L^4} &= \Big\| \int_0^t P_N S_{(t-s)} F(x^N(s)) ds \Big\|_{L^4} \\ &\le \int_0^t \Big\| P_N S_{\frac{(t-s)}{2}} \Big\|_{L\left(L^2((0,1),\mathbb{R}), L^4((0,1),\mathbb{R})\right)} \cdot \Big\| S_{\frac{(t-s)}{2}} F(x^N(s)) \Big\|_{L^2} ds\end{aligned}$$

for every $t \in [0, \tau]$. Due to Lemma 5.8, we obtain

$$\begin{aligned}\big\|y^N(t)\big\|_{L^4} &\le \int_0^t \left(\tfrac{t-s}{2}\right)^{-\frac{1}{8}} \cdot \Big\| S_{\frac{(t-s)}{2}} F(x^N(s)) \Big\|_{L^2} ds \\ &= 2^{\frac{1}{8}} |c| \int_0^t (t-s)^{-\frac{1}{8}} \cdot \Big\| S_{\frac{(t-s)}{2}} \left[\left((x^N(s))^2 \right)' \right] \Big\|_{L^2} ds \\ &\le 2^{\frac{1}{8}} |c| \int_0^t (t-s)^{-\frac{1}{8}} \cdot 4 \cdot (T+1) \cdot \left(\tfrac{t-s}{2}\right)^{-\frac{3}{4}} \cdot \|(x^N(s))^2\|_{L^1} ds \\ &= 2^{\frac{7}{8}} 4(T+1)|c| \int_0^t (t-s)^{-\frac{7}{8}} \cdot \|x^N(s)\|^2_{L^2} ds\end{aligned}$$

and

$$\big\|y^N(t)\big\|_{L^4} \le 8(T+1)|c| \int_0^t (t-s)^{-\frac{7}{8}} \cdot \big\|x^N(s)\big\|^2_{L^2} ds \quad \le 64(T+1) T^{\frac{1}{8}} |c| \sup_{0 \le s \le \tau} \|x^N(s)\|^2_{L^2} \tag{5.11}$$

for every $t \in [0, \tau]$. Additionally, we have

$$\begin{aligned}&\big\|y^N(t)\big\|_{C([0,1],\mathbb{R})} = \Big\| \int_0^t \left(P_N S_{(t-s)} F(x^N(s)) \right) ds \Big\|_{C([0,1],\mathbb{R})} \\ &\le \int_0^t \Big\| P_N S_{\frac{(t-s)}{2}} \Big\|_{L\left(L^2((0,1),\mathbb{R}), C([0,1],\mathbb{R})\right)} \cdot \Big\| S_{\frac{(t-s)}{2}} \Big\|_{L\left(H^{-1}((0,1),\mathbb{R}), L^2((0,1),\mathbb{R})\right)} \cdot \big\| F(x^N(s)) \big\|_{H^{-1}} ds\end{aligned}$$

and

$$\begin{aligned}\big\|y^N(t)\big\|_{C([0,1],\mathbb{R})} &\le \int_0^t \left(\tfrac{t-s}{2}\right)^{-\frac{1}{4}} \cdot \left(\tfrac{t-s}{2}\right)^{-\frac{1}{2}} \cdot \big\| F(x^N(s)) \big\|_{H^{-1}} ds \\ &= 2^{\frac{3}{4}} |c| \int_0^t (t-s)^{-\frac{3}{4}} \cdot \Big\| \frac{\partial}{\partial x} (x^N(s))^2 \Big\|_{H^{-1}} ds \\ &\le 2|c| \int_0^t (t-s)^{-\frac{3}{4}} \cdot \big\| (x^N(s))^2 \big\|_{L^2} ds\end{aligned}$$

for every $t \in [0, \tau]$ due to Lemma 5.8. This implies

$$\|y^N(t)\|_{C([0,1],\mathbb{R})} \le 2|c| \int_0^t (t-s)^{-\frac{3}{4}} \cdot \|x^N(s)\|^2_{L^4} ds = 8|c| T^{\frac{1}{4}} \sup_{0 \le s \le \tau} \|x^N(s)\|^2_{L^4} \tag{5.12}$$

for every $t \in [0, \tau]$. Combining (5.10), (5.11) and (5.12) yields

$$\begin{aligned}
\sup_{0\le t\le\tau} \|y^N(t)\|_{C([0,1],\mathbb{R})} &\le 8|c|T^{\frac{1}{4}}\Big(\sup_{0\le t\le\tau}\|y^N(t)\|_{L^4} + \sup_{0\le t\le\tau}\|o^N(t)\|_{L^4}\Big)^2 \\
&\le 8|c|T^{\frac{1}{4}}\Big(64(T+1)T^{\frac{1}{8}}|c|\sup_{0\le s\le\tau}\|x^N(s)\|^2_{L^2} + z\Big)^2 \\
&\le 2^{16}(|c|^3+1)(T+1)^3\Big(\sup_{0\le s\le\tau}\|x^N(s)\|^2_{L^2} + z^2\Big)
\end{aligned}$$

and therefore

$$\begin{aligned}
\sup_{0\le t\le\tau} \|y^N(t)\|_{C([0,1],\mathbb{R})} &\le 2^{16}(|c|^3+1)(T+1)^3\Big(\Big(\sup_{0\le s\le\tau}\|y^N(s)\|_{L^2} + z\Big)^2 + z^2\Big) \\
&\le 2^{17}(|c|^3+1)(T+1)^3\Big(\sup_{0\le s\le\tau}\|y^N(s)\|^2_{L^2} + 2z^2\Big) \\
&\le 2^{17}(|c|^3+1)(T+1)^3\left(\exp\{4(c^2+1)(T+1)z^2\} + 2z^2\right) .
\end{aligned}$$

Finally,

$$\begin{aligned}
\sup_{0\le t\le\tau} \|y^N(t)\|_{C([0,1],\mathbb{R})} &\le 2^{18}(|c|^3+1)(T+1)^3\exp\{6(c^2+1)(T+1)z^2\} \\
&\le e^{13}\cdot\exp\{3(c^2+1)(T+1)\}\cdot\exp\{6(c^2+1)(T+1)z^2\} \\
&\le \exp\{13(c^2+1)(T+1)(z^2+1)\} ,
\end{aligned}$$

where $z \in \mathbb{R}$ is given by $z := \sup_{0\le t\le\tau}\|o^N(t)\|_{C([0,1],\mathbb{R})}$. ☐

LEMMA 5.8. *Let $S : (0,T] \to L\big(H^{-1}((0,1),\mathbb{R}), C([0,1],\mathbb{R})\big)$ be given by Lemma 4.6 and let $P_N : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R}), N \in \mathbb{N}$, be given by (4.11). Then, we obtain*

$$\|P_N\, S_t\|_{L\left(L^2((0,1),\mathbb{R}),C([0,1],\mathbb{R})\right)} \le t^{-\frac{1}{4}}, \qquad \|P_N\, S_t\|_{L\left(L^2((0,1),\mathbb{R}),L^4((0,1),\mathbb{R})\right)} \le t^{-\frac{1}{8}}$$

and

$$\|S_t\|_{L\left(H^{-1}((0,1),\mathbb{R}),L^2((0,1),\mathbb{R})\right)} \le t^{-\frac{1}{2}}, \qquad \|S_t(v')\|_{L^2} \le 4\,(T+1)\,t^{-\frac{3}{4}}\|v\|_{L^1}$$

for every $t \in (0,T], N \in \mathbb{N}$ and every $v \in C^1([0,1],\mathbb{R})$.

Proof. [Proof of Lemma 5.8] First of all, we have

$$\begin{aligned}
\|P_N S_t v\|_{C([0,1],\mathbb{R})} &= \sup_{0\le x\le 1}\Big|\sum_{n=1}^N 2e^{-n^2\pi^2 t}\int_0^1 \sin(n\pi s)\,v(s)\,ds\cdot\sin(n\pi x)\Big| \\
&\le \sum_{n=1}^N 2e^{-n^2\pi^2 t}\Big|\int_0^1 \sin(n\pi s)\,v(s)\,ds\Big| \\
&\le \Big(\sum_{n=1}^N 2e^{-2n^2\pi^2 t}\Big)^{\frac{1}{2}}\Big(\sum_{n=1}^N 2\Big|\int_0^1 \sin(n\pi s)\,v(s)\,ds\Big|^2\Big)^{\frac{1}{2}} \\
&\le \Big(\int_0^\infty \frac{2}{2\pi\sqrt{t}}e^{-\frac{x^2}{2}}\,dx\Big)^{\frac{1}{2}}\|v\|_{L^2} \le \left(\tfrac{2}{\pi t}\right)^{\frac{1}{4}}\|v\|_{L^2} \le t^{-\frac{1}{4}}\|v\|_{L^2} \quad (5.13)
\end{aligned}$$

for every $t \in (0, T], N \in \mathbb{N}$ and every $v \in L^2((0,1), \mathbb{R})$, which shows the first estimate. Moreover, we obtain

$$\|P_N S_t v\|_{L^4}^4 = \int_0^1 |P_N S_t v(x)|^4 \, dx \leq \|P_N S_t v\|_{L^2}^2 \|P_N S_t v\|_{C([0,1],\mathbb{R})}^2 \leq t^{-\frac{1}{2}} \|v\|_{L^2}^4$$

for every $v \in L^2((0,1), \mathbb{R})$, $t \in (0, T]$ and every $N \in \mathbb{N}$ due to (5.13) for the second estimate. For the third estimate we obtain

$$\begin{aligned}
\|S_t w\|_{L^2}^2 &= \sum_{n=1}^{\infty} \Big(\sqrt{2}\, e^{-n^2\pi^2 t}\, w(\sin(n\pi(\cdot))) \Big)^2 \\
&= \frac{t^{-1}}{2} \sum_{n=1}^{\infty} 2n^2\pi^2 t e^{-2n^2\pi^2 t} \frac{|w(\sqrt{2}\sin(n\pi(\cdot)))|^2}{n^2\pi^2} \leq \frac{t^{-1}}{2} \|w\|_{H^{-1}}^2 \leq t^{-1} \|w\|_{H^{-1}}^2
\end{aligned}$$

for every $t \in (0, T], w \in H^{-1}((0,1), \mathbb{R})$, since $x \cdot e^{-x} \leq 1$ for every $x \in \mathbb{R}$. Finally, we establish the fourth estimate. To this end we consider

$$(S_t w)(x) = \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t} \int_0^1 \sin(n\pi s)\, w(s)\, ds \cdot \sin(n\pi x)$$

and

$$(S_t w)'(x) = \sum_{n=1}^{\infty} 2n\pi e^{-n^2\pi^2 t} \int_0^1 \sin(n\pi s)\, w(s)\, ds \cdot \cos(n\pi x)$$

for every $t \in (0, T], x \in [0, 1]$ and every $w \in L^2((0,1), \mathbb{R})$. This implies

$$\begin{aligned}
\|(S_t w)'\|_{C([0,1],\mathbb{R})} &\leq \sum_{n=1}^{\infty} 2n\pi e^{-n^2\pi^2 t} \Big| \int_0^1 \sin(n\pi s)\, w(s)\, ds \Big| \\
&\leq \Big(\sum_{n=1}^{\infty} \Big(\sqrt{2} n\pi e^{-n^2\pi^2 t} \Big)^2 \Big)^{\frac{1}{2}} \cdot \Big(\sum_{n=1}^{\infty} \Big| \int_0^1 \sqrt{2} \sin(n\pi s)\, w(s)\, ds \Big|^2 \Big)^{\frac{1}{2}} \\
&= \Big(\sum_{n=1}^{\infty} 2n^2\pi^2 e^{-2n^2\pi^2 t} \Big)^{\frac{1}{2}} \cdot \|w\|_{L^2} \\
&\leq \Big(\int_0^{\infty} 2(x+1)^2 \pi^2 e^{-2x^2\pi^2 t} \, dx \Big)^{\frac{1}{2}} \cdot \|w\|_{L^2}
\end{aligned}$$

and

$$\begin{aligned}
\|(S_t w)'\|_{C([0,1],\mathbb{R})} &\leq \Big(\int_0^{\infty} 2\pi^2 (x^2 + 2x + 1) e^{-2x^2\pi^2 t} \, dx \Big)^{\frac{1}{2}} \cdot \|w\|_{L^2} \\
&\leq 2\pi \Big(\int_0^{\infty} (x^2 + 1) e^{-2x^2\pi^2 t} \, dx \Big)^{\frac{1}{2}} \cdot \|w\|_{L^2} \\
&= \frac{2\pi}{(2\pi t)^{\frac{1}{4}}} \Big(\int_0^{\infty} \frac{1}{\sqrt{2\pi}} \Big(\frac{x^2}{4\pi^2 t} + 1 \Big) e^{-\frac{x^2}{2}} \, dx \Big)^{\frac{1}{2}} \cdot \|w\|_{L^2} \\
&\leq \frac{(2\pi)^{\frac{3}{4}}}{t^{\frac{1}{4}}} \Big(1 + \frac{1}{4\pi^2 t} \Big)^{\frac{1}{2}} \cdot \|w\|_{L^2} \leq 4(T+1)^{\frac{1}{2}} t^{-\frac{3}{4}} \cdot \|w\|_{L^2}
\end{aligned}$$

for every $t \in (0,T]$ and every $w \in L^2((0,1),\mathbb{R})$. Therefore, we obtain

$$
\begin{aligned}
\|S_t(v')\|_{L^2} &= \sup_{\|w\|_{L^2}\le 1} |\langle w, S_t(v')\rangle_{L^2}| = \sup_{\|w\|_{L^2}\le 1} |\langle S_t w, v'\rangle_{L^2}| = \sup_{\|w\|_{L^2}\le 1} |\langle (S_t w)', v\rangle_{L^2}| \\
&\le \sup_{\|w\|_{L^2}\le 1} \|(S_t w)'\|_{C([0,1],\mathbb{R})} \cdot \|v\|_{L^1} \le 4(T+1)t^{-\frac{3}{4}} \|v\|_{L^1}
\end{aligned}
$$

for every $t \in (0,T]$ and every continuously differentiable function $v : [0,1] \to \mathbb{R}$. □

LEMMA 5.9. *Let $F : C([0,1],\mathbb{R}) \to H^{-1}((0,1),\mathbb{R})$ be given by Lemma 4.8. Then, we have*

$$\langle v, v'' + F(v+w)\rangle_{L^2} \le 2\,c^2 \|v\|_{L^2}^2 \|w\|_{C([0,1],\mathbb{R})}^2 + 2\,c^2 \|w\|_{C([0,1],\mathbb{R})}^4$$

for all twice continuously differentiable functions $v : [0,1] \to \mathbb{R}$ and $w : [0,1] \to \mathbb{R}$ with $v(0) = v(1) = 0$ and where $c \in \mathbb{R}$ is used in Lemma 4.8.

Proof. [Proof of Lemma 5.9] We have

$$
\begin{aligned}
\langle v, F(v+w)\rangle_{L^2} &= c \int_0^1 v \cdot ((v+w)^2)'\,dx \\
&= -c \int_0^1 v' \cdot v^2\,dx - 2c \int_0^1 v' \cdot v \cdot w\,dx - c \int_0^1 v' \cdot w^2\,dx
\end{aligned}
$$

and therefore as $\int_0^1 v' v^2 dx = 0$

$$\langle v, F(v+w)\rangle_{L^2} = -c \int_0^1 v' \cdot v^2\,dx - 2c \int_0^1 v' \cdot v \cdot w\,dx - c \int_0^1 v' \cdot w^2\,dx$$

for all continuously differentiable functions $v, w : [0,1] \to \mathbb{R}$ with $v(0) = v(1) = 0$. Hence, we obtain

$$
\begin{aligned}
\langle v, F(v+w)\rangle_{L^2} &\le 2|c| \cdot \|v'\|_{L^2} \cdot \|v\|_{L^2} \cdot \|w\|_{C([0,1],\mathbb{R})} + |c| \cdot \|v'\|_{L^1} \cdot \|w\|_{C([0,1],\mathbb{R})}^2 \\
&\le 2|c| \left(\|v\|_{L^2} \cdot \|w\|_{C([0,1],\mathbb{R})} + \|w\|_{C([0,1],\mathbb{R})}^2 \right) \cdot \|v'\|_{L^2} \\
&\le |c|^2 \left(\|v\|_{L^2} \cdot \|w\|_{C([0,1],\mathbb{R})} + \|w\|_{C([0,1],\mathbb{R})}^2 \right)^2 + \|v'\|_{L^2}^2
\end{aligned}
$$

and finally

$$\langle v, F(v+w)\rangle_{L^2} \le 2c^2 \|v\|_{L^2}^2 \|w\|_{C([0,1],\mathbb{R})}^2 + 2c^2 \|w\|_{C([0,1],\mathbb{R})}^4 + \|v'\|_{L^2}^2$$

for all continuously differentiable functions $v, w : [0,1] \to \mathbb{R}$ with $v(0) = v(1) = 0$. Therefore, we obtain

$$
\begin{aligned}
\langle v, v'' + F(v+w)\rangle_{L^2} &= -\|v'\|_{L^2}^2 + \langle v, F(v+w)\rangle_{L^2} \\
&\le 2c^2 \|v\|_{L^2}^2 \|w\|_{C([0,1],\mathbb{R})}^2 + 2c^2 \|w\|_{C([0,1],\mathbb{R})}^4
\end{aligned}
$$

for all twice continuously differentiable functions $v : [0,1] \to \mathbb{R}$ and $w : [0,1] \to \mathbb{R}$ with $v(0) = v(1) = 0$, which is the assertion. □

5.7. Gronwall's Lemma. In the proof of Theorem 3.1 the following Gronwall inequality is needed. It is very similar to Lemma 7.1.11 in [13] and just for completeness its proof is presented below.

LEMMA 5.10. *Let $b, \beta \in (0, \infty)$, $a \in [0, \infty)$ and let $e : [0, T] \to [0, \infty)$ be a $\mathcal{B}([0,T])/\mathcal{B}([0,\infty))$-measurable mapping, which satisfies*

$$e(t) \leq a + b \int_0^t (t-s)^{(\beta-1)}\, e(s)\, ds < \infty \tag{5.14}$$

for every $t \in [0,T]$. Then, we obtain

$$e(t) \leq a \cdot \mathrm{E}_\beta \left(t (b\, \Gamma(\beta))^{\frac{1}{\beta}} \right) \tag{5.15}$$

for every $t \in [0, T]$, where $\mathrm{E}_\beta : [0,\infty) \to [0,\infty)$ is given by $\mathrm{E}_\beta(x) = \sum_{n=0}^{\infty} \frac{x^{(n\beta)}}{\Gamma(n\beta+1)}$ for every $x \in [0,\infty)$.

Proof. [Proof of Lemma 5.10] Consider the set

$$U := \Big\{ u : [0,T] \to [0,\infty) \text{ is } \mathcal{B}([0,T])/\mathcal{B}([0,\infty))\text{-measurable} \;\Big|\; \int_0^t (t-s)^{(\beta-1)} u(s)\, ds < \infty \;\forall\, t \in [0,T] \Big\}$$

and the mapping

$$B : U \to U, \qquad (Bu)(t) = b \int_0^t (t-s)^{(\beta-1)}\, u(s)\, ds$$

for every $t \in [0,T]$ and every $u \in U$. First of all, B is well defined, since

$$\begin{aligned}
\int_0^t (t-s)^{(\beta-1)} (Bu)(s)\, ds &= \int_0^t (t-s)^{\beta-1} b \int_0^s (s-r)^{\beta-1} u(r)\, dr ds \\
&= b \int_0^t \int_0^s (t-s)^{\beta-1}(s-r)^{\beta-1} u(r)\, dr\, ds \\
&= b \int_0^t \int_r^t (t-s)^{\beta-1}(s-r)^{\beta-1} ds\, u(r)\, dr
\end{aligned}$$

and therefore

$$\begin{aligned}
\int_0^t (t-s)^{\beta-1} (Bu)(s)\, ds &= b \int_0^t \int_0^{(t-r)} (t-r-s)^{\beta-1} s^{\beta-1}\, ds\, u(r)\, dr \\
&= b \int_0^t (t-r)^{2\beta-1} \int_0^1 (1-s)^{\beta-1} s^{\beta-1}\, ds\, u(r)\, dr \\
&= \frac{b\Gamma(\beta)^2}{\Gamma(2\beta)} \int_0^t (t-r)^{2\beta-1} u(r)\, dr \\
&\leq \frac{b\Gamma(\beta)^2 T^\beta}{\Gamma(2\beta)} \int_0^t (t-r)^{\beta-1} u(r)\, dr < \infty
\end{aligned}$$

for every $t \in [0,T]$ and every $u \in U$. Moreover, we have

$$(B^n u)(t) = \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)} \int_0^t (t-s)^{n\beta-1} u(s)\, ds \tag{5.16}$$

for every $t \in [0, T]$, $u \in U$ and every $n \in \mathbb{N}$. We show (5.16) by induction on $n \in \mathbb{N}$. In the case $n = 1$ equation (5.16) follows from the definition of $B : U \to U$. Therefore, we assume that (5.16) holds for a fixed $n \in \mathbb{N}$ and obtain

$$\begin{aligned}\left(B^{(n+1)}u\right)(t) &= b\int_0^t (t-s)^{\beta-1}(B^n u)(s)\,ds \\ &= \frac{b^{n+1}\Gamma(\beta)^n}{\Gamma(n\beta)}\int_0^t\int_0^s (t-s)^{\beta-1}(s-r)^{n\beta-1}u(r)\,dr\,ds \\ &= \frac{b^{n+1}\Gamma(\beta)^n}{\Gamma(n\beta)}\int_0^t\int_r^t (t-s)^{\beta-1}(s-r)^{n\beta-1}ds\,u(r)\,dr\end{aligned}$$

and

$$\begin{aligned}\left(B^{(n+1)}u\right)(t) &= \frac{b^{n+1}\Gamma(\beta)^n}{\Gamma(n\beta)}\int_0^t\int_0^{(t-r)} (t-r-s)^{\beta-1}s^{n\beta-1}\,ds\,u(r)\,dr \\ &= \frac{b^{n+1}\Gamma(\beta)^n}{\Gamma(n\beta)}\int_0^t (t-r)^{(n+1)\beta-1}\int_0^1 (1-s)^{\beta-1}s^{n\beta-1}\,ds\,u(r)\,dr \\ &= \frac{b^{n+1}\Gamma(\beta)^n}{\Gamma(n\beta)}\int_0^t (t-r)^{(n+1)\beta-1}\frac{\Gamma(\beta)\cdot\Gamma(n\beta)}{\Gamma((n+1)\beta)}u(r)\,dr \\ &= \frac{(b\,\Gamma(\beta))^{n+1}}{\Gamma((n+1)\beta)}\int_0^t (t-r)^{2\beta-1}u(r)\,dr\end{aligned}$$

for every $t \in [0, T]$ and every $u \in U$. This shows (5.16) for every $n \in \mathbb{N}$ by induction. Additionally, we have

$$e(t) \le a + (Be)(t)$$

for every $t \in [0, T]$ due to (5.14). Moreover, note that $(Bu)(t) \le (Bv)(t)$ for every $t \in [0, T]$ and every $u, v \in U$ with $u(t) \le v(t)$ for all $t \in [0, T]$. Hence, we obtain

$$(Be)(t) \le (B(a + Be))(t) = (Ba)(t) + (B^2 e)(t)$$

and therefore

$$e(t) \le a + (Be)(t) \le a + (Ba)(t) + (B^2 e)(t)$$

for every $t \in [0, T]$. Iterating this idea yields

$$\begin{aligned}e(t) &\le a + (Ba)(t) + (B^2 a)(t) + \ldots + (B^{(n-1)}a)(t) + (B^n e)(t) \\ &= a + \sum_{k=1}^{n-1}\frac{(b\,\Gamma(\beta))^k}{\Gamma(k\beta)}\int_0^t (t-s)^{k\beta-1}a\,ds + (B^n e)(t) \\ &= a + \sum_{k=1}^{n-1} a\frac{(b\,\Gamma(\beta))^k}{\Gamma(k\beta)}\int_0^t s^{k\beta-1}\,ds + (B^n e)(t)\end{aligned}$$

and

$$\begin{aligned}e(t) &\le a + \sum_{k=1}^{n-1} a\cdot\frac{(b\,\Gamma(\beta))^k}{\Gamma(k\beta)}\cdot\frac{t^{k\beta}}{k\beta} + (B^n e)(t) \\ &= a\cdot\sum_{k=0}^{n-1}\frac{\left(t(b\,\Gamma(\beta))^{\frac{1}{\beta}}\right)^{k\beta}}{\Gamma(k\beta+1)} + (B^n e)(t)\end{aligned}$$

for every $t \in [0, T]$ and every $n \in \mathbb{N}$ due to (5.16). Since $\lim_{n\to\infty}(B^n e)(t) = 0$ for every $t \in [0, T]$, we finally obtain the assertion by taking the limit $n \to \infty$. $\square$

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REFERENCES

[1] Alabert, A., and Gyöngy, I. On numerical approximation of stochastic Burgers' equation. In *From stochastic calculus to mathematical finance*. Springer, Berlin, 2006, pp. 1–15.
[2] Blömker, D., Flandoli, F., and Romito, M. Markovianity and ergodicity for a surface growth PDE. *Ann. Probab. 37*, 1 (2009), 275–313.
[3] Blömker, D., Maier-Paape, S., and Schneider, G. The stochastic Landau equation as an amplitude equation. *Discrete Contin. Dyn. Syst. Ser. B 1*, 4 (2001), 527–541.
[4] Da Prato, G., and Debussche, A. Stochastic Cahn-Hilliard equation. *Nonlinear Anal., Theory Methods Appl. 26*, 2 (1996), 241–263.
[5] Da Prato, G., Debussche, A., and Temam, R. Stochastic Burgers' equation. *NoDEA, Nonlinear Differ. Equ. Appl. 1*, 4 (1994), 389–402.
[6] Da Prato, G., and Zabczyk, J. *Stochastic equations in infinite dimensions*, vol. 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.
[7] Da Prato, G., and Zabczyk, J. *Ergodicity for infinite-dimensional systems*, vol. 229 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1996.
[8] Flandoli, F., and Gatarek, D. Martingale and stationary solutions for stochastic Navier-Stokes equations. *Probab. Theory Relat. Fields 102*, 3 (1995), 367–391.
[9] Flandoli, F., and Romito, M. Markov selections for the 3D stochastic Navier-Stokes equations. *Probab. Theory Relat. Fields 140*, 3-4 (2008), 407–458.
[10] Grecksch, W., and Kloeden, P. E. Time-discretised Galerkin approximations of parabolic stochastic PDEs. *Bull. Austral. Math. Soc. 54*, 1 (1996), 79–85.
[11] Gyöngy, I. Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. II. *Potential Anal. 11*, 1 (1999), 1–37.
[12] Gyöngy, I., and Nualart, D. On the stochastic Burgers' equation in the real line. *Ann. Probab. 27*, 2 (1999), 782–802.
[13] Henry, D. *Geometric theory of semilinear parabolic equations*, vol. 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.
[14] Jentzen, A. Higher order pathwise numerical approximation of spdes with additive noise. 34. submitted.
[15] Jentzen, A. Pathwise Numerical Approximations of SPDEs with Additive Noise under Non-global Lipschitz Coefficients. *Potential Analysis* (2009). (published online).
[16] Jentzen, A. *Taylor Expansions for Stochastic Partial Differential Equations*. Johann Wolfgang Goethe-University, Frankfurt am Main, Germany, 2009. Dissertation.
[17] Jentzen, A., and Kloeden, P. E. Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 465*, 2102 (2009), 649–667.
[18] Jentzen, A., Kloeden, P. E., and Neuenkirch, A. Pathwise approximation of stochastic differential equations on domains: higher order convergence rates without global Lipschitz coefficients. *Numer. Math. 112*, 1 (2009), 41–64.
[19] Kloeden, P. E., Lord, G. J., Neuenkirch, A., and Shardlow, T. The exponential integrator scheme for stochastic partial differential equations: pathwise error bounds. submitted.
[20] Kloeden, P. E., and Shott, S. Linear-implicit strong schemes for Itô-Galerkin approximations of stochastic PDEs. *J. Appl. Math. Stochastic Anal. 14*, 1 (2001), 47–53. Special issue: Advances in applied stochastics.
[21] Liu, D. Convergence of the spectral method for stochastic Ginzburg-Landau equation driven by space-time white noise. *Commun. Math. Sci. 1*, 2 (2003), 361–375.
[22] Lord, G. J., and Rougemont, J. A numerical scheme for stochastic PDEs with Gevrey regularity. *IMA J. Numer. Anal. 24*, 4 (2004), 587–604.
[23] Lord, G. J., and Shardlow, T. Postprocessing for stochastic parabolic partial differential equations. *SIAM J. Numer. Anal. 45*, 2 (2007), 870–889 (electronic).
[24] Müller-Gronbach, T., Ritter, K., and Wagner, T. Optimal pointwise approximation of a linear stochastic heat equation with additive space-time white noise. In *Monte Carlo and*

quasi-Monte Carlo methods 2006. Springer, Berlin, 2007, pp. 577–589.
[25] Müller-Gronbach, T., Ritter, K., and Wagner, T. Optimal pointwise approximation of infinite-dimensional Ornstein-Uhlenbeck processes. *Stoch. Dyn. 8*, 3 (2008), 519–541.
[26] Pettersson, R., and Signahl, M. Numerical approximation for a white noise driven SPDE with locally bounded drift. *Potential Anal. 22*, 4 (2005), 375–393.
[27] Prévôt, C., and Röckner, M. *A concise course on stochastic partial differential equations*, vol. 1905 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.
[28] Robinson, J. C. *Infinite-dimensional dynamical systems*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2001. An introduction to dissipative parabolic PDEs and the theory of global attractors.
[29] Runst, T., and Sickel, W. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, vol. 3 of *de Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, 1996.
[30] Sell, G. R., and You, Y. *Dynamics of evolutionary equations*, vol. 143 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2002.